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Modeling diverse expectations in an aggregated New Keynesian Model ☆

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ABSTRACT

We explore a New Keynesian Model with diverse beliefs and study the aggregation problems in the log-linearized economy. We show the solution of these problems depend upon the belief structure. Agents' beliefs are described by individual state variables and satisfy three Rationality Axioms, leading to the emergence of an aggregate state variable named "mean market state of belief." In equilibrium, endogenous variables are functions of mean market belief and this state variable is the tool used to solve the aggregation problems.

Diverse beliefs alter the problem faced by a central bank since the source of fluctuations is not only exogenous shocks but also market expectations. Due to diverse beliefs the effects of policy instruments are not monotonic and the trade-off between inflation and output volatilities is complex. Also, monetary policy can counter the effects of market belief by aggressive anti-inflation policy but at the cost of increased volatility of financial markets and individual consumption.

1. Introduction

The New Keynesian Model (in short NKM) has become an important tool of macroeconomics. Due to its assumption of monopolistic competition, prices are firms' strategic variables and price stickiness is a cause for money non-neutrality and

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efficacy of monetary policy. The model is inherently heterogenous: it does not start with a representative agent and the large number of household-firms need not be identical. Up to now most research on the NKM has been done under the strong Rational Expectations (in short RE) assumption that all agents are identical and with policy implications that may be questioned. It is thus only natural to ask what is the effect of heterogeneity on the conduct of monetary policy and the exploration of this question is our long term goal. In this paper we focus on the narrow question of how to formulate an aggregate model when agents hold heterogenous expectations. To that end we formulate a *microeconomic* NKM in which agents hold diverse beliefs and investigate whether the model can be aggregated. “Aggregation” means that we deduce from the microeconomic equilibrium, in a manner compatible with the probabilistic structure of agents' beliefs, *a set of structural relations among macroeconomic aggregates* that constitute a dynamic macroeconomic NKM. Moreover, if such aggregation is possible, what are the implications of diverse beliefs to the resulting macroeconomic dynamics and policy?

Before proceeding we note that as the era of RE comes to a close, it is useful to keep in mind two points. First, the success of RE in disciplining macroeconomic modeling should not obscure the fact that the term “rational” is merely a label. Rationality of actions and rationality of beliefs have little to do with each other and using the term “rational” in RE has tended to brand all other beliefs as “irrational.” Rational agents who hold diverse beliefs do not satisfy the RE requirements but may satisfy other plausible principles of rationality. Indeed, the study of axioms of belief rationality is a fruitful area of research that can fill the wide open space between the extremities of RE and true irrational beliefs.

A second point relates to private information. Many scholars use the device of asymmetric private information as the “cause” of diverse beliefs. Indeed, some view diverse beliefs as *equivalent* to asymmetric information. This is theoretically and empirically the wrong solution and Kurz (2008, 2009) explains why. Suffices to say that market behavior of agents holding diverse beliefs with common information is very different from the case when they have private information. Under private information individuals guard their private information and deduce private information from prices. Without private information agents are willing to reveal their forecasts and use the opinions of others (i.e. market belief) only to forecast future prices and other endogenous variables, not as a source from which to deduce information they do not have. In addition, all empirical evidence associates diverse forecasts to diverse modeling or diverse interpretation of public information (e.g. Batchelor and Dua, 1991; Frankel and Froot, 1990; Frankel and Rose, 1995; Kandel and Pearson, 1995; Takagi, 1991). Finally, the volatility of RE models with private information is fully determined by exogenous shocks, consequently they cannot deliver the main dynamic implications of economies with rational and diverse beliefs with common information (see Kurz, 2009). This key implication is that diverse beliefs constitute a volatility amplification mechanism and excess economic fluctuations are *caused* by diverse beliefs. It is an economic risk which is generated *within the economy*, not by exogenous shocks, and is thus called *Endogenous Uncertainty* (See Kurz and Wu, 1996; Kurz, 1997). These properties are explored in Kurz (2009, 2011a) and discussed later in Section 4.

To explore problems of aggregation, we concentrate on the standard version of the NKM. With this in mind we follow developments in Woodford (2003), Walsh (2010) and Gali (2008). We note the axiomatic approach of Branch and McGough (2009) to the aggregation problem, a method adopted by others such as Branch and Evans (2006, 2011) and Branch and McGough (2011). The Branch and McGough's (2009) axioms are made directly on the expectation operators, not on beliefs. As they are motivated by bounded rationality, they violate typical models with diverse beliefs. In contrast, we specify *rationality* axioms on beliefs and show they offer a natural route to a NKM with diverse beliefs where aggregation is attained in the log linearized economy. This last point is important since it will be clear a “representative household” does not exist in the model developed below and aggregation of the true economy is not possible in most cases. Instead, we study the aggregation problem in the log linear economy which is the standard economy used for virtually any policy analysis.

The source of our aggregation results is the structure of agents' beliefs. To highlight this point note there is a growing literature on monetary policy with diverse beliefs which treats the problem of aggregation as follows. For any model developed denote the model's expectations of x_{t+1} by $E_t x_{t+1}$ and suppose that in the model there are N types of agents in proportions n_t^i, each holding properly specified conditional expectation $E_t^i x_{t+1}$. All members of the same type hold the same expectations. Then, it is assumed that

$$E_t x_{t+1} = \sum_{i=1}^{N} n_t^i [E_t^i x_{t+1}]$$

where the aggregate expectations $E_t x_{t+1}$ is assumed a conditional expectation with respect to some probability. Examples for this approach are Adam (2007), Anufriev et al. (2008), Massaro (2012), Arifovic et al. (2007), Brazier et al. (2008) and De Grauwe (2011). We first note that it is well known (see Kurz, 2008) that as defined above $E_t x_{t+1}$ violates iterated expectations and, in general, there is no probability measure with respect to which it is a conditional expectation. Averaging probability measures does not yield a regular probability measure. Going beyond this technical issue, we show in this paper that the problem of aggregation is deeply connected to the problem of defining a concept of market belief. Hence, in order for the agents' optimal decision functions to aggregate one must impose restrictions on the individual beliefs underlying the expectations $E_t^i x_{t+1}$ which cannot be arbitrary, as assumed above. We shall show in this paper that it is exactly the Rationality Axioms of individual beliefs that provide a set of sufficient conditions for aggregation to be attained.

Ideas about diverse beliefs we use here are drawn from the literature on the Rational Beliefs (in short RB) theory. Kurz (1994, 1997) are early work and Kurz (2009, 2011) are recent surveys. The work here extends results of Kurz (2008) and Kurz and Motolese (2011). As to monetary policy, Motolese (2001, 2003) shows that diverse beliefs cause, on their own, money non-neutrality. Kurz et al. (2005) and Jin (2007) offer the first formal models showing diverse beliefs constitute an independent cause for business cycle fluctuations and model calibration that reproduces the data of the US economy. In the same spirit Branch and McGough (2011) and De Grauwe (2011) show that boundedly rational diverse beliefs amplify business cycle fluctuations. Other approaches to the problem include Lorenzoni (2009) and Milani (2011).

We have already noted (with regard to the definition of market expectations) the growing interest in the impact of diverse beliefs on monetary policy. Now we add several more comments. The Kurz et al. (2005) model assumes agents hold diverse beliefs and that prices are fully flexible, yet the model exhibits money non-neutrality, showing that sticky prices offer only one of the routes to efficacy of monetary policy. They investigate the ability of different monetary policy rules to stabilize fluctuations caused primarily by diverse beliefs. Hence, belief diversity is a volatility amplification mechanism which, in turn, becomes the object of monetary stabilization policy. Woodford (2010) explores the impact of "Near Rational Expectations" on optimal monetary policy. Other non RE papers that study efficacy of policy, approach it from the perspective of learning. Howitt (1992) uses a standard macroeconomic model and shows instability under learning of interest rate pegging and related rules. Similarly, Bullard and Mitra (2002) show that if agents follow adaptive learning, the stability of the Taylor-type rules is questionable. Evans and Honkapohja (2003, 2006) study a NKM with a representative agent but with non RE belief due to learning. They study the joint stability of the economy and learning and show convergence to RE under stability of learning. They assume agents are boundedly rational as they do not know the equilibrium map and make forecasts based only on their learning model. Here we assume agents are rational and know the equilibrium map but have diverse beliefs about the state variables of the system.

What are the paper's results? Note first that the present paper is the first NKM which is studied from an RB perspective. Sections 2 and 3 develop the model and explore the problem of aggregating the log linearized economy, leading to intermediate results which clarify the basic problems that need to be solved. In Section 4 we develop the theory of belief formation, extending ideas in Kurz (2008) and Kurz and Motolese (2011), by specifying three Axioms which must be satisfied by the belief of a rational agents, and we explain why belief diversity is compatible with the Axioms. These rationality conditions have two implications which are central to this paper. First, they are the basis for macroeconomic dynamics in the sense that they imply volatility amplification and hence Endogenous Uncertainty. Second, we show that under the assumed structure of belief, aggregation is possible and leads to a consistent macroeconomic model. However, the aggregate model has key parameters deduced from the microeconomic equilibrium and which, in turn, *depend upon the policy parameters*. Hence, if a policy rule is changed, new equilibrium parameters need to be derived from the micro-equilibrium and hence the macro-model itself changes. Hence, the study of feasible stabilization of a policy rule entails a study of the rule's impact on the parameters of the macroeconomic model it induces. This process of evaluation is entirely absent from the standard macroeconomic model based on the representative agent.

Section 5 explores the properties of the microeconomic equilibrium which is at the basis of the aggregated NKM with diverse beliefs developed in this paper. Section 6 provides a brief example, via simulations, of the impact of diverse beliefs on the efficacy of monetary policy. Understanding the results requires a clarification of what the central bank aims to stabilize when agents hold diverse beliefs. A standard Real Business Cycles (RBC) model assumes that technology shocks (to be defined later) have a standard deviation of 0.0072 deduced from the Solow residual, a practice that has been universally rejected, leading to a consensus that the true standard deviation is much smaller. Our starting point is therefore a value of 0.004 assigned to this standard deviation with the implications that much of the model's volatility is due to the effect of expectations and other shocks. This is a change in the problem faced by a central bank since one conclusion of this paper is that an efficient policy rule depends upon the nature of the shocks and the cause of volatility. Comparisons between the results of this paper with standard results in the literature show that the central bank has different tasks in the two models: in a standard RBC model under RE the driving force is a large technology shock (perhaps with other exogenous shocks) whose effect the bank aims to stabilize due to sticky prices. In the models of this paper the technology shocks are small hence economic fluctuations are driven by modest exogenous shocks but amplified substantially by market expectations, making central bank policy concerned with stabilizing the effect of market expectations on volatility.

Volatility effects of expectations are present even in models with flexible prices; hence a central bank must stabilize the volatility of actual output level (or its deviation from steady state) rather than the volatility of the gap between output and the output level at the flexible price equilibrium. Section 6.1 provides details on why the gap is not an object of central bank stabilization in an economy with diverse beliefs.

As to stabilization, we study the outcomes of policy rules defined on inflation and output (or expected inflation and output discussed briefly in Section 7), with weights ξ_π and ξ_y, respectively. Under a standard RE formulation of technology shocks the response is monotonic in the two instruments: output volatility falls with ξ_y and rises with ξ_π while inflation volatility rises with ξ_y and falls with ξ_π. Hence, a central bank faces a policy choice between volatility of output and inflation. With diverse beliefs and other exogenous shocks these results do not hold and Section 6 provides an example of such results. In a later paper we shall carry out a detailed simulation study of the impact of diverse beliefs on the efficacy of monetary policy (see Kurz, 2012 for a preliminary version). It will examine in detail the impact of diverse beliefs on the trade-off between volatility of output and inflation. The example in Section 6 shows the following:

- Under diverse beliefs the effect of policy instruments (ξ_y, ξ_π) is not monotonic, consequently there is a limited policy trade-off between volatility of aggregate output and volatility of inflation. Trade-off may be between regions of the policy space rather than on a smooth differentiable surface.
- Monetary policy can counter the effects of market expectations on the volatility of output and inflation by aggressive anti-inflation ξ_π policy which stabilizes both (σ_y, σ_π), but at a cost.

- Aggressive choice of entails the cost of fluctuating interest rates, volatile financial markets and high volatility of individual consumption. Hence, although aggressive anti-inflation policy can stabilize the aggregates, a central bank may avoid such a policy due to a concern over the volatility of financial markets and individual consumption. An efficient policy is moderate.

2. Household j's problem and Euler equations

The standard formulation starts with a continuum of agents and products but this formulation is not natural when one draws a random sample of the order of the continuum. Hence, although in the development below we write integrals for mean values, it is natural to think of such integrals as arising in a large economy when one takes limits of means as sample size increases to infinity.

Household j is a producer-consumer that produces intermediate commodity j at price p_{jt} with production technology which uses only labor (without capital) defined by

$$Y_{jt} = \zeta_t N_{jt}, \zeta_t > 0 \text{ a random variable with } E^m(\zeta_t) = 1$$

We explain later what the probability measure m is. The household solves a maximization problem with a penalty on excessive borrowing and lending of the form

$$\text{Max } E_t^j \sum_{\tau=0}^{\infty} \beta^\tau \left(\frac{1}{1-\sigma}(C^j_{t+\tau})^{1-\sigma} - \frac{1}{1+\eta}(L^j_{t+\tau})^{1+\eta} + \frac{1}{1-b}\left(\frac{M^j_{t+\tau}}{P_{t+\tau}}\right)^{1-b} - \frac{\tilde{\tau}_b}{2}\left(\frac{B^j_{t+\tau}}{P_{t+\tau}}\right)^2 \right), \; 0<\beta<1, \, \sigma>0, \, \eta>0, \, b>0. \tag{1}$$

The penalty replaces an institutional constraint to limit borrowing. We set $\tilde{\tau}_b$ very small (typically $\tilde{\tau}_b < 10^{-4}$) to replace transversality conditions and define a solution with explosive borrowing to be a non-equilibrium. The budget constraint, with transfers used for redistribution to be explained below, is defined by

$$C^j_t + \frac{M^j_t}{P_t} + \frac{B^j_t}{P_t} + \frac{T^j_t}{P_t} = \left(\frac{W_t}{P_t}\right) L^j_t + \left[\frac{B^j_{t-1}(1+r_{t-1}) + M^j_{t-1}}{P_{t-1}}\right]\left(\frac{P_{t-1}}{P_t}\right) + \frac{1}{P_t}\left[p_{jt}Y_{jt} - W_t N_{jt}\right] \tag{2}$$

(M^j_0, B^j_0) is given, all j. Initial aggregate debt is 0 and aggregate money supply at $t=0$ is given.

C is consumption, M is money holding, L is labor supplied, T are transfers, W is nominal wage, B are bond holdings and r is a nominal interest rule defined later as a function of aggregate variables. Equilibrium real balances, inflation rate and nominal interest rate will then determine the equilibrium price level.

The standard Euler equations are as follows. Optimum with respect to bond purchases B^j_t is

$$\tilde{\tau}_b\left(\frac{B^j_t}{P_t}\right) + (C^j_t)^{-\sigma} = E^j_t\left[\beta\left(C^j_{t+1}\right)^{-\sigma} \frac{1+r_t}{(P_{t+1}/P_t)}\right]. \tag{3a}$$

Optimum with respect to labor is

$$(C^j_t)^{-\sigma}\left(\frac{W_t}{P_t}\right) = (L^j_t)^\eta \tag{3b}$$

and optimum with respect to money is

$$1 - \frac{(M^j_t/P_t)^{-b}}{(C^j_t)^{-\sigma}} = E^j_t\left[\beta\left(\frac{C^j_{t+1}}{C^j_t}\right)^{-\sigma} \frac{1}{(P_{t+1}/P_t)}\right]. \tag{3c}$$

Eq. (3a)–(3c) implies that the demand for money is determined by the following condition:

$$\frac{(M^j_t/P_t)^{-b}}{(C^j_t)^{-\sigma}} = \frac{r_t}{1+r_t} - \frac{\tilde{\tau}_b}{1+r_t}\left(\frac{B^j_t}{P_t(C^j_t)^{-\sigma}}\right). \tag{4}$$

We proceed as in a cashless economy by ignoring (4) and how the central bank provides liquidity to satisfy the demand for money in (4) via the agent's transfers. The central bank sets the nominal interest rate.

We now log linearize the Euler equations. If X has a riskless steady state $\overline{X}$ then the notation is $\hat{x}_t = (X_t - \overline{X})/\overline{X}$ except for borrowing when $\hat{b}_t = B_t/(P_t\overline{Y})$ with zero steady state value. Our log linear approximations assumes zero inflation steady state hence we let $\overline{\pi} = 1, log(P_t/P_{t-1}) \simeq \hat{\pi}_t = \hat{p}_t - \hat{p}_{t-1}$ and

$$\hat{c}^j_t = E^j_t(\hat{c}^j_{t+1}) - \left(\frac{1}{\sigma}\right)\left[\hat{r}_t - E^j_t(\hat{\pi}_{t+1})\right] + \tau_b \hat{b}^j_t, \tau_b = \frac{\tilde{\tau}_B}{\sigma}\overline{Y}^{1+\sigma}, \; \tau_b < 10^{-4}, \tag{5a}$$

$$-\sigma(\hat{c}^j_t) + (\hat{w}_t - \hat{p}_t) = \eta(\hat{\ell}^j_t). \tag{5b}$$

In steady state $\overline{C}^j = \overline{Y}^j = \overline{C} = \overline{Y}, \overline{\pi} = 1, \overline{L}^j = \overline{N}^j = \overline{Y}$. The final term $\tau_b \hat{b}^j_t$ imposes j's transversality conditions which insists on bounded borrowing. Observe that (5b) aggregates and equilibrium conditions $\int_0^1 \hat{c}^j_t dj = \hat{c}_t = \hat{y}_t = \int_0^1 \hat{y}^j_t \, dj$,

$\int_0^1 \hat{n}_t^j \, dj = \hat{n}_t = \hat{\ell}_t = \int_0^1 \hat{\ell}_t^j \, dj$ imply the important relation

$$(\hat{w}_t - \hat{p}_t) = \eta(\hat{n}_t) + \sigma(\hat{y}_t). \tag{5b$'$}$$

On the other hand, (5a) does not aggregate since it entails an expression of the form $\int_0^1 E_t^j(\hat{c}_{t+1}^j)dj$ or in the finite case $1/N\sum_{j=1}^N E_t^j(\hat{c}_{t+1}^j)$.

Average individuals' forecasts of the deviation of their future consumption from steady state is computable number but is not a natural macroeconomic aggregate. For this reason we first rewrite (5a) as

$$\hat{c}_t^j = E_t^j(\hat{c}_{t+1}) + (E_t^j(\hat{c}_{t+1}^j) - E_t^j(\hat{c}_{t+1})) - \left(\frac{1}{\sigma}\right)\left[\hat{r}_t - E_t^j(\hat{\pi}_{t+1})\right] + \tau_b \hat{b}_t^j. \tag{5a$'$}$$

Next, introduce

Definition 1. $\overline{E}_t = \int_0^1 E_t^j \, dj$ means: for any random variable x, $\overline{E}_t(x) = \int_0^1 E_t^j(x)dj$.

Average agents' diverse probabilities are not a proper probability and the operator $\overline{E}_t$ is not a conditional expectation deduced from a probability measure (see Kurz, 2008). It is an average forecast and does not obey the law of iterated expectations. Since $\hat{c}_t = \hat{y}_t$, $\hat{b}_t = 0$ averaging (5a$'$) leads to

$$\hat{y}_t = \overline{E}_t(\hat{y}_{t+1}) + \int_0^1 \left(E_t^j(\hat{c}_{t+1}^j) - E_t^j(\hat{c}_{t+1})\right)\mathrm{d}j - \left(\frac{1}{\sigma}\right)\left[\hat{r}_t - \overline{E}_t(\hat{\pi}_{t+1})\right] \tag{6}$$

Individual penalties vanish while the middle term does not aggregate. It occurs when mean agents' forecasts of *own* consumption differ from mean forecast of *mean* consumption. In (6) we use the definition

$$\Phi_t(\hat{c}) = \int_0^1 (E_t^j(\hat{c}_{t+1}^j) - E_t^j(\hat{c}_{t+1}))dj \tag{6a}$$

Proposition 1. *Under diverse beliefs the IS curve in a log linearized economy is defined by* (6)–(6a)

$$\hat{y}_t = \overline{E}_t(\hat{y}_{t+1}) + \Phi_t(\hat{c}) - \left(\frac{1}{\sigma}\right)\left[\hat{r}_t - \overline{E}_t(\hat{\pi}_{t+1})\right] \tag{7}$$

where the term $\Phi_t(\hat{c})$ *is not directly aggregated. It reflects the structure of market belief.*

Diverse beliefs has thus a dual impact on (7): the mean forecast operator $\overline{E}_t$ which violates the law of iterated expectations and the term $\Phi_t(\hat{c})$. Under RE and representative household $E_t^j(\hat{c}_{t+1}^j) = \overline{E}_t(\hat{c}_{t+1})$ and the extra terms disappear. These terms are natural to diverse beliefs hence pivotal issues to be examined.

3. Demand functions and optimal pricing under monopolistic competition

We adopt a standard model of household-producer-monopolistic competitor with the Calvo (1983) model for sticky prices hence our development is familiar. There is a large number (perhaps a continuum or, equivalently, a large N) of products and each agent produces one product which is substitutable with all others. Final consumption of household j is constructed from intermediate outputs as follows:

$$C_t^j = \left[\int_0^1 (C_{it}^j)^{\frac{\theta-1}{\theta}} \, di\right]^{\frac{\theta}{\theta-1}}, \quad \theta > 1$$

At price p_{it} consumption cost is $\int_0^1 p_{it} C_{it}^j \, di$. Minimizing cost subject to $C_t^j \le \left[\int_0^1 (c_{it}^j)^{\frac{\theta-1}{\theta}} \, di\right]^{\frac{\theta}{\theta-1}}$ leads to

$$C_{it}^j = \left(\frac{p_{it}}{P_t}\right)^{-\theta} C_t^j \tag{8}$$

P_t is price of final consumption, which is the price level. Equilibrium in the final goods market requires

$$P_t \equiv \left[\int_0^1 p_{it}^{1-\theta} \, di\right]^{\frac{1}{1-\theta}} \tag{8a}$$

Aggregate (8) over households j to obtain the market demand function for intermediate commodity i, given aggregate consumption. But aggregate consumption equals aggregate income. Hence, considering j who produces intermediate good j, the demand for firm's j product is defined by

$$Y_{jt}^d = \left(\frac{p_{jt}}{P_t}\right)^{-\theta} Y_t \tag{8b}$$

with implied required labor input of

$$N_{jt} = \frac{1}{\zeta_t}\left(\frac{p_{jt}}{P_t}\right)^{-\theta} Y_t$$

with probability $(1-\omega)$ a firm adjusts prices at each date, independently over time.

Assumption 1. In a Calvo model firms with diverse beliefs select different optimal prices. Assume that the sample of firms allowed to adjust prices at each date is selected in dependently across agents hence the distribution of agents in terms of output or belief is the same whether one looks at those who adjust prices or those who do not adjust prices.

We now examine the price level in (8a). At t a random sample is taken as a set of firms S_t in [0,1] of measure $1-\omega$ that adjust prices at t and S_t^c in [0,1] of measure ω that do not adjust. By the key Assumption 1 the mean price of those firms that do not change price equals the date $t-1$ price hence

$$P_t^{1-\theta} = \int_{S_t} p_{jt}^{*(1-\theta)}\, dj + \int_{S_t^c} p_{j,t-1}^{(1-\theta)}\, dj = \int_{S_t} p_{jt}^{*(1-\theta)}\, dj + \omega P_{t-1}^{1-\theta}$$

p_{jt}^* is the optimal price of j hence,

$$1 = \int_{S_t}\left(\frac{p_{jt}^*}{P_t}\right)^{(1-\theta)} dj + \omega\left(\frac{P_{t-1}}{P_t}\right)^{1-\theta}. \tag{9}$$

Define $q_{jt}^* = p_{jt}^*/P_t$ and log linearize (9) to conclude the equation $0 = \int_{S_t}\hat{q}_{jt}^*\, dj - \omega\hat{\pi}_t$. Hence we have

$$\int_{S_t}\hat{q}_{jt}^*\, dj = \omega\hat{\pi}_t. \tag{10a}$$

At steady state $\overline{p} = \overline{p}$ and using notation $\Delta X_t = X_t - \overline{X}$, it follows from (9) that a log linearization leads to;

$$\int_{S_t^c}\Delta\left(\frac{p_{j,t-1}}{P_t}\right) dj = -\omega\hat{\pi}_t \tag{10b}$$

By Assumption 1, with probability 1, (10a) is independent of sets S_t. The distributions of characteristics are the same in all random sets and (10a) changes only by change in state variables of the economy. If every firm selects its optimal price, the mean over the population is related to (10a) through the relation

$$\int_{S_t}\hat{q}_{jt}^*\, dj = (1-\omega)\int_0^1 \hat{q}_{jt}^*\, dj \Rightarrow \int_0^1 \hat{q}_{jt}^*\, dj = \frac{\omega}{1-\omega}\hat{\pi}_t.$$

Marginal cost: Since $Y_{jt} = \zeta_t N_{jt}$ variable cost function of j is $W_t(Y_{jt}/\zeta_t)$. Nominal marginal cost is W_t/ζ_t and real marginal cost is $\varphi_t = (1/\zeta_t)(W_t/P_t)$. Deviations from steady state are therefore $\hat{\varphi}_t = -\hat{\zeta}_t + \hat{w}_t - \hat{p}_t$.

Since agent j is a monopolistic competitor, maximizing (1) with respect to output is the same as maximizing with respect to p_{jt}. In the next section we use the demand function to define the profits function:

$$\Pi_{jt} = \frac{1}{P_t}\left[p_{jt}Y_{jt} - W_t N_{jt}\right] = \left[\frac{p_{jt}}{P_t} - \frac{1}{\zeta_t}\frac{W_t}{P_t}\right]Y_{jt} = \left[\left(\frac{p_{jt}}{P_t}\right)^{1-\theta} - \frac{1}{\zeta_t}\frac{W_t}{P_t}\left(\frac{p_{jt}}{P_t}\right)^{-\theta}\right]Y_t \tag{11}$$

We now turn to optimal pricing. Agent j owns firm j and manages its business. His optimal pricing is selected by maximizing (1) subject to (2) and (11) together with the Calvo type price limitation.

Insurance and anonymity Assumption 2:. An agent-firm chooses an optimal price subject to the budget constraint (2) and (11) and *considers the transfer as a lump sum*. However, the actual level of transfers made ensures all firms have the same real profits. Hence, transfers to firm j equal.

$$\frac{T_t^j}{P_t} = \Pi_t - \Pi_t^j, \quad \Pi_t = \int_0^1 \Pi_t^j\, dt.$$

Discussion:. Assumption 2 removes all income effects of random price adjustments. It is equivalent to assuming either that profits are insured or that all agents-firms have equal ownership share in all firms but agent-firm j manages firm j by selecting an optimal price so as to maximize (1) subject to (2). Anonymity means here that agent-firm j assumes it is small and has no effect on the transfers it receives or pays.

Profit in (11) requires j to select optimal price to maximize (1) subject to the budget constraint at all future dates $(t+\tau)$ in which, with probability ω^τ, the firm cannot change the price at t. The budget is

$$C_{t+\tau}^j + \frac{M_{t+\tau}^j}{P_{t+\tau}} + \frac{B_{t+\tau}^j}{P_{t+\tau}} + \frac{T_{t+\tau}^j}{P_{t+\tau}}$$

$$= \left(\frac{W_{t+\tau}}{P_{t+\tau}}\right) L^j_{t+\tau} + \left[\frac{B^j_{t+\tau-1}(1+r_{t+\tau-1}) + M^j_{t+\tau-1}}{P_{t+\tau-1}}\right]\left(\frac{P_{t+\tau-1}}{P_{t+\tau}}\right) + \left[\frac{p_{jt}}{P_{t+\tau}} - \frac{1}{\zeta_{t+\tau}}\frac{W_{t+\tau}}{P_{t+\tau}}\right]\left(\frac{p_{jt}}{P_{t+\tau}}\right)^{-\theta} Y_{t+\tau}.$$

Now, the first order conditions apply only to terms involving p^*_{jt} and these conditions are

$$E^j_t \sum_{\tau=0}^{\infty} \beta^\tau \omega^\tau (C^j_{t+\tau})^{-\sigma} \left((1-\theta)\left(\frac{p^*_{jt}}{P_{t+\tau}}\right)^{-\theta} + \theta\varphi_{t+\tau}\left(\frac{p^*_{jt}}{P_{t+\tau}}\right)^{-\theta-1}\right)\frac{Y_{t+\tau}}{P_{t+\tau}} = 0$$

where $\varphi_{t+\tau} = (1/\zeta_{t+\tau})(W_{t+\tau}/P_{t+\tau})$. Using (8b) this condition is equivalent to

$$\left[E^j_t \sum_{t=0}^{\infty} \beta^\tau \omega^\tau (C^j_{t+\tau})^{-\sigma} Y_{t+\tau}\left(\frac{P_t}{P_{t+\tau}}\right)^{-\theta}\left((1-\theta)\left(\frac{p^*_{jt}}{P_t}\right)\frac{P_t}{P_{t+\tau}} + \theta\varphi_{t+\tau}\right)\right]\frac{1}{p^*_{jt}}\left(\frac{p^*_{jt}}{P_t}\right)^{-\theta} = 0$$

Cancel the end terms and solve for (p^*_{jt}/P_t) to deduce the optimal price of a firm that adjusts price at date t

$$\left(\frac{p^*_{jt}}{P_t}\right) = \left(\frac{\theta}{\theta-1}\right)\frac{E^j_t \sum_{\tau=0}^{\infty} \beta^\tau \omega^\tau (C^j_{t+\tau})^{-\sigma} Y_{t+\tau}\varphi_{t+\tau}(\frac{P_{t+\tau}}{P_t})^{\theta}}{E^j_t \sum_{\tau=0}^{\infty} \beta^\tau \omega^\tau (C^j_{t+\tau})^{-\sigma} Y_{t+\tau}(\frac{P_{t+\tau}}{P_t})^{\theta-1}} \tag{12}$$

Aiming to aggregate (12) we log linearize it as follows. First write it as

$$\sum_{\tau=0}^{\infty} \beta^\tau \omega^\tau E^j_t\left[(C^j_{t+\tau})^{-\sigma} Y_{t+\tau}\left(\frac{P_{t+\tau}}{P_t}\right)^{\theta-1}\right] q^*_{jt} = \left(\frac{\theta}{\theta-1}\right)\sum_{\tau=0}^{\infty} \beta^\tau \omega^\tau E^j_t\left[(C^j_{t+\tau})^{-\sigma} Y_{t+\tau}\varphi_{t+\tau}\left(\frac{P_{t+\tau}}{P_t}\right)^{\theta}\right].$$

Log linearization of the left hand side around the riskless steady state yields

$$\frac{(\overline{C}^j)^{1-\sigma}}{(1-\beta\omega)} + \frac{(\overline{C}^j)^{1-\sigma}}{(1-\beta\omega)}\hat{q}^*_{jt} + (\overline{C}^j)^{1-\sigma}\sum_{\tau=0}^{\infty} \beta^\tau \omega^\tau E^j_t[-\sigma(\hat{c}^j_{t+\tau}) + \hat{y}_{t+\tau} + (\theta-1)(\hat{p}_{t+\tau} - \hat{p}_t)]$$

and the right hand side

$$\left(\frac{\theta}{\theta-1}\right)\left(\frac{(\overline{C}^j)^{1-\sigma}}{(1-\beta\omega)}\overline{\varphi} + \overline{\varphi}(\overline{C}^j)^{1-\sigma}\sum_{\tau=0}^{\infty} \beta^\tau \omega^\tau E^j_t[\hat{\varphi}_{t+\tau} - \sigma(\hat{c}^j_{t+\tau}) + \hat{y}_{t+\tau} + \theta(\hat{p}_{t+\tau} - \hat{p}_t)]\right).$$

Equalizing both note two facts. First, in the steady state prices are flexible and it is well known that

$$\left(\frac{\theta}{\theta-1}\right)\overline{\varphi} = 1$$

Second, when equalizing the two sides all terms involving $(\overline{C}^j)^{1-\sigma}$, $E^j_t(\hat{c}^j_{t+\tau})$ and $\hat{y}_{t+\tau}$ cancel and we have

$$\frac{\hat{q}^*_{jt}}{(1-\beta\omega)} = \sum_{\tau=0}^{\infty} \beta^\tau \omega^\tau E^j_t[\hat{\varphi}_{t+\tau} + (\hat{p}_{t+\tau} - \hat{p}_t)] = \sum_{\tau=0}^{\infty} \beta^\tau \omega^\tau E^j_t[\hat{\varphi}_{t+\tau} + \hat{p}_{t+\tau}] - \frac{1}{(1-\beta\omega)}\hat{p}_t \tag{13}$$

Eq. (13) shows the only difference among firms that adjust prices arises due to difference in expectations of economy wide variables. No j specific variable appears on the right side. From (13) one deduces that

$$\hat{q}^*_{jt} + \hat{p}_t = (1-\beta\omega)[\hat{\varphi}_t + \hat{p}_t] + \beta\omega(1-\beta\omega)E^j_t\left(\sum_{\tau=0}^{\infty} \beta^\tau \omega^\tau E^j_{t+1}[\hat{\varphi}_{t+1+\tau} + \hat{p}_{t+1+\tau}]\right).$$

It leads to a relation between optimal price at t and expected optimal price at $t+1$ if j can adjust price at $t+1$:

$$\hat{q}^*_{jt} + \hat{p}_t = (1-\beta\omega)[\hat{\varphi}_t + \hat{p}_t] + \beta\omega E^j_t[\hat{q}^*_{j(t+1)} + \hat{p}_{t+1}]$$

or

$$\hat{q}^*_{jt} = (1-\beta\omega)\hat{\varphi}_t + \beta\omega E^j_t[\hat{q}^*_{j(t+1)} + \hat{\pi}_{t+1}] \Rightarrow \int_0^1 \hat{q}^*_{jt} = (1-\beta\omega)\hat{\varphi}_t + (\beta\omega)\int_0^1 E^j_t[\hat{q}^*_{j(t+1)} + \hat{\pi}_{t+1}]dj. \tag{14}$$

Introduce the notation:

$$\hat{q}_t = \int_0^1 \hat{q}^*_{jt}, \quad \Phi_t(\hat{q}) = \int_0^1 (E^j_t\hat{q}^*_{j(t+1)} - E^j_t\hat{q}_{(t+1)})dj. \tag{14a}$$

$\Phi_t(\hat{q})$ is analogous to $\Phi_t(\hat{c})$ in (6a) and both are not aggregate variables. Using (14a) we have

$$\hat{q}_t = (1-\beta\omega)\hat{\varphi}_t + (\beta\omega)\overline{E}_t(\hat{q}_{t+1} + \hat{\pi}_{t+1}) + (\beta\omega)\Phi_t(\hat{q}) \tag{15}$$

Now recall that $\hat{q}_t = (\omega/(1-\omega))\hat{\pi}_t$ hence (15) can be written as

$$\hat{\pi}_t = \frac{(1-\beta\omega)(1-\omega)}{\omega}\hat{\varphi}_t + \beta\overline{E}_t\hat{\pi}_{t+1} + \beta(1-\omega)\Phi_t(\hat{q}) \tag{15a}$$

This last term leads to the second basic proposition

Proposition 2. *The forward looking Phillips Curve in the log linearized economy depends upon the market distribution of beliefs and takes the general form.*

$$\hat{\pi}_t = \kappa\hat{\varphi}_t + \beta\overline{E}_t\hat{\pi}_{t+1} + \beta(1-\omega)\Phi_t(\hat{q}), \quad \kappa = \frac{(1-\beta\omega)(1-\omega)}{\omega}, \quad \varphi_t = \frac{1}{\zeta_t}\frac{W_t}{P_t}.$$

Diverse beliefs are expressed via the mean operator $\overline{E}$ and the extra term $\Phi_t(\hat{q})$.

From the definition of marginal cost $\hat{\varphi}_t = -\hat{\zeta}_t + (\hat{w}_t - \hat{p}_t)$ and from the first order condition for labor (5b′) we have $(\hat{w}_t - \hat{p}_t) = \eta(\hat{n}_t) + \sigma(\hat{y}_t)$. Hence $\hat{\varphi}_t = -\hat{\zeta}_t + \eta\hat{n}_t + \sigma\hat{y}_t$. But from the production function we also have that $\hat{n}_t = \hat{y}_t - \hat{\zeta}_t$ hence we finally have that

$$\hat{\varphi}_t = -\hat{\zeta}_t + \eta[\hat{y}_t + \hat{\zeta}_t] + \sigma\hat{y}_t = -(1+\eta)\hat{\zeta}_t + (\eta+\sigma)\hat{y}_t$$

We can then rewrite the Phillips Curve as

$$\hat{\pi}_t = -\kappa(1+\eta)\hat{\zeta}_t + \kappa(\eta+\sigma)\hat{y}_t + \beta\overline{E}_t\hat{\pi}_{t+1} + \beta(1-\omega)\Phi_t(\hat{q})$$

This is a forward looking Phillips Curve except that now average expectations are not of the representative household but rather, of the diverse beliefs in the market.

3.1. Intermediate summary of the system

Suppose the monetary rule is $\hat{r}_t = \xi_\pi\hat{\pi}_t + \xi_y\hat{y}_t + u_t$ where u_t measures random variability in the central bank's application of the rule, reflecting bank's judgment or error in special circumstances. We then have

$$\text{IS curve} \quad \hat{y}_t = \overline{E}_t(\hat{y}_{t+1}) + \Phi_t(\hat{c}) - \left(\frac{1}{\sigma}\right)\left[\hat{r}_t - \overline{E}_t(\hat{\pi}_{t+1})\right] \tag{16a}$$

$$\text{Phillips curve} \quad \hat{\pi}_t = \kappa(\eta+\sigma)\hat{y}_t + \beta\overline{E}_t\hat{\pi}_{t+1} + \beta(1-\omega)\Phi_t(\hat{q}) - \kappa(1+\eta)\hat{\zeta}_t \tag{16b}$$

$$\text{Monetary rule} \quad \hat{r}_t = \xi_\pi\hat{\pi}_t + \xi_y\hat{y}_t + u_t \tag{16c}$$

This is a New Keynesian system with three endogenous variables and two exogenous shocks: a *technology supply shock* and a *bank's random policy shock*[1], with two differences from standard models. First, the extra non-aggregate terms $(\Phi_t(\hat{c}), \Phi_t(\hat{q}))$. Second, expectations are not based on a single probability measure and the operator $\overline{E}_t$ violates iterated expectations. It is merely the average date t conditional forecast. Such averaging among correlated random variables introduces a new economic volatility which is not present in standard models, and hence it needs to be explored. The construction of a macroeconomic model depends upon the structure of market beliefs.

4. Beliefs

For beliefs to be diverse there must be something agents do not know and on which they disagree. Here we stipulate it to be the distribution of the exogenous shocks $(\hat{\zeta}_t, u_t)$ but exogenous shocks vary in the literature. The true process of technology and bank's policy shocks is not known. It is a non-stationary process, subject to structural changes and regime shifts[2]. Following the RB approach (see Kurz, 1994, 1997), agents have past data on these variables hence the empirical distribution of the shocks is common knowledge. By "empirical distribution" we mean the distribution one computes from a long series of observations by computing relative frequencies or moments *of all past data together* and where such computations are made without judgment or attempts to estimate the effect of any transitory short term events. Computation of the empirical distribution of a stochastic process leads to the formulation of a stationary probability on sequences which is then common knowledge to all agents and plays a crucial role in the theory developed here. We denote this stationary probability with the letter m and refer to it as the "empirical distribution" or the "empirical probability." To simplify assume that $(\hat{\zeta}_t, u_t)$ have a Markov distribution with empirical transitions which are Markov of the form

$$\hat{\zeta}_{t+1} = \lambda_\zeta\hat{\zeta}_t + \rho^\zeta_{t+1} \tag{17a}$$

$$u_{t+1} = \lambda_u u_t + \rho^u_{t+1} \tag{17b}$$

[1] Many macro-models introduce shocks without specifying their microeconomic origin. Since the policy shock u enters only through the nominal rate, it is equivalent to any shock which is restricted only to the IS curve. Hence, we view the policy shock as a proxy for any shock which is restricted only to the IS curve.

[2] In our view economic growth consists of a sequence of eras with different products, technologies and institutions. Although it is common to think of these as "regime shifts," transition from one era to the next is often smooth and slow, with few discontinuous jumps. Markov transition functions which then describe such changes in the text are then just averages over many different structures. For more details see Kurz (2009).

$$\begin{pmatrix} \rho^{\zeta}_{t+1} \\ \rho^{u}_{t+1} \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \sigma^2_{\zeta}, & 0, \\ 0, & \sigma^2_u \end{bmatrix} = \Sigma \right), \quad \text{i.i.d}$$

The truth is that both processes are subject to shifts in structure, taking the true form.

$$\hat{\zeta}_{t+1} = \lambda_{\zeta}\hat{\zeta}_t + \lambda^s_{\zeta} s_t + \tilde{\rho}^{\zeta}_{t+1} \tag{18a}$$

$$u_{t+1} = \lambda_u u_t + \lambda^s_u s_t + \tilde{\rho}^u_{t+1} \tag{18b}$$

$$\begin{pmatrix} \tilde{\rho}^{\zeta}_{t+1} \\ \tilde{\rho}^{u}_{t+1} \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \tilde{\sigma}^2_{\zeta}, & 0, \\ 0, & \tilde{\sigma}^2_u \end{bmatrix} \right)$$

The parameters s_t are unobserved hence (17a) and (17b) are time averages of (18a) and (18b), hence the sequence of s_t has a zero mean. To simplify we assume there is only one factor and there will be only one belief parameter to pin down a belief about *all state variables*. More general models have multiple factors. We aim to discuss an approach to belief formation that applies to a wide family of models. In some applications we examine specific examples of models with only one exogenous shock, in which case we assume $u_t = 0$.

4.1. Describing belief with state variables: rationality and belief diversity imply dynamics

Agents may believe (17a) and (17b) are the true transitions, and some do, but typically they do not and form their own beliefs about these structural parameters. We introduce agent i's state variable *denoted by* g^i_t and used to describe i's belief. It is a perception variable which pins down his subjective transition functions *of all state variables*. Agent i knows g^i_t but since forecast samples are taken, he observes the *distribution* of g^j_t across j but not *specific* g^j_t of others. This entails a small measure of information asymmetry as each agent knows his own g^i_t but only the distribution of the others. This asymmetry does not matter since we also assume “anonymity.” It means agent i is small and does not assume g^i_t impacts market belief. For a proper expression of anonymity suppose for a moment the economy has finite agents with a distribution $(g^1_t, g^2_t, \ldots, g^N_t)$ of individual beliefs. Anonymity means an agent perceives the distribution of market belief to be given hence agent j does not associate g^j_t with himself. This problem does not arise here since, due to log-linear approximation, only the mean of the g^j_t has a market impact and we denote it by $Z_t = \int_{[0,1]} g^j_t \, dj$ where the letter Z is used instead $\overline{g}_t$ of. To simplify further we assume agents observe only frequency distributions of the, g^j_t not individual g^j_t hence all past distributions are common knowledge.

How is g^i_t used by an agent? We use the notation of $(\hat{\zeta}^i_{t+1}, u^i_{t+1})$.[3] to express i's *perception* of $t+1$ shocks and $(E^i_t\hat{\zeta}_{t+1}, E^i_t u_{t+1})$ are among them. It specifies the *difference* between date t forecast and the forecasts under the empirical probability m. Agent i's date t *perceived* distribution of $(\hat{\zeta}^i_{t+1}, u^i_{t+1})$ is

$$\hat{\zeta}^i_{t+\tau} = \lambda_{\zeta}\hat{\zeta}_t + \lambda^g_{\zeta} g^i_t + \rho^{i\zeta}_{t+1} \tag{19a}$$

$$u^i_{t+1} = \lambda_u u_t + \lambda^g_u g^i_t + \rho^{iu}_{t+1} \tag{19b}$$

$$\begin{pmatrix} \rho^{i\zeta}_{t+1} \\ \rho^{iu}_{t+1} \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \hat{\sigma}^2_{\zeta}, & \hat{\sigma}_{\zeta u}, \\ \hat{\sigma}_{\zeta u}, & \hat{\sigma}^2_u \end{bmatrix} \right)$$

The assumption that $(\hat{\sigma}^2_{\zeta}, \hat{\sigma}_{\zeta u}, \hat{\sigma}^2_u)$ are the same for all agents is made for simplicity. It follows that given public information I_t at date t, g^i_t measures the difference.

$$E^i[(\hat{\zeta}_{t+1}, u_{t+1}) | I_t, g^i_t] - E^m[(\hat{\zeta}_{t+1}, u_{t+1}) | I_t] = (\lambda^g_{\zeta} g^i_t, \lambda^g_u g^i_t). \tag{20}$$

We adopt two rationality principles.

Rationality Principle 1: A belief cannot be a *constant transition* unless an agent believes the stationary transition (17a) and (17b) is the truth.

Rationality Principle 2: A belief does not deviate from (17a) and (17b) consistently and hence the belief index g^i_t must have an unconditional mean of zero.

Condition (20) shows how g^i_t is measured using forecast data since $E^m[(\hat{\zeta}_{t+1}, u_{t+1}) | I_t]$ is a standard econometric forecast employing past data by making no judgment about special circumstances on any time interval. When $g^i_t = 0$, $\rho^{i\zeta}_{t+1} = \rho^{\zeta}_{t+1}$, $\rho^{iu}_{t+1}, = \rho^u_{t+1}$ agent i believes m is the truth. Since beliefs are about changes in society, g^i_t reflect belief about different economies. For example, in 1900 the g^i_t are about electricity and combustion engines, but in 2000 they reflected beliefs about information technology.

[3] The notation $(\hat{\zeta}^i_{t+1}, u^i_{t+1})$ is used to highlight the *perception* of the macro-variables $(\hat{\zeta}_{t+1}, u_{t+1})$ by agent i before the variables are observed. Hence, when one uses the expectations of a macro-variable $\boldsymbol{x}_{t+1}$ by agent i, there is no difference between $E^i_t x^i_{t+1}$ and $E^i_t x_{t+1}$. Hence, for any variable x^i_{t+1} is the *perception* of x_{t+1} by agent i before it is observed, and $E^i_t x_{t+1}$ is the expectations of x_{t+1} by i, *in accordance with his perception*. We stress that perception is defined only before a variable is observed. This procedure does not apply to i-specific variables such as $E^i_t \hat{c}^i_{t+1}$ which has a natural interpretation.

The two rationality principles imply that if an economy has diverse beliefs and such diversity persists without opinions tending to merge, then a typical agent's belief g_t^i *must fluctuate over time*. This is the most important implication of rationality requirements: *rationality implies dynamics*. The reason is simple. Agents cannot hold constant, invariant, transitions unless they are (17a) and (17b). Since it is a well established empirical fact that belief diversity persists on most economic variables, (17a) and (17b) are not the belief of most, but since the time average of an agent's transitions must be (17a) and (17b), they must fluctuate.[4] This relation between rationality and dynamics is central to the RB approach (e.g. Kurz, 1994, 1996, 2009). The natural next step is the treatment of belief dynamics as state variables. Since beliefs fluctuate, such time changes of transition functions may be fixed by an agent in advance for the infinite future. More typically they are random and unknown as they may depend upon assessments made, data observed and signals received in the future. Since the first two principles do not specify the dynamics of belief, the third principle addresses the issue. To keep things simple we state it and prove it only with respect to one observed exogenous shock which, as an example, is chosen here to be $\hat{\zeta}_t$.

Rationality Principle 3: The transition functions of g_t^i are Markov, taking a form which exhibits persistence and if $u_t=0$ the form is

$$g_{t+1}^i = \lambda_Z g_t^i + \lambda_Z^{\zeta}[\hat{\zeta}_{t+1} - \lambda_{\zeta}\hat{\zeta}_t] + \rho_{t+1}^{ig}, \quad \rho_{t+1}^{ig} \sim N(0, \sigma_g^2) \tag{21}$$

where ρ_{t+1}^{ig} are correlated across *i*. Correlation of ρ_{t+1}^{ig} reflects correlated beliefs across agents and this correlation is a crucial component of the theory. Analogous law of motion applies if the shock is only u_t or both u_t and $\hat{\zeta}_t$.

Rationality Principle 3 says date $t+1$ agent belief state is unknown at t but has a Markov transition. It is analogous to the concept of a "type" in games with incomplete information where an agent type is revealed only in the future. We use the term "forecasting belief" in the sense of taking expectations of objects like (21) or its aggregate and *uncertainty of future belief state is central to this theory*. How can one justify (21) which plays such a key role in the theory? The first answer is that the data supports this specification (see Kurz and Motolese, 2011). Alternatively, we prove (21) analytically as a result of Bayesian rationality.

4.2. *Deducing (21) from a model of Bayesian rationality*[5]

In standard Bayesian inference an agent observes data generated by a stationary process with an unknown *fixed* parameter. He starts with a prior on the parameter and uses Bayesian inference for retrospective updating of his belief. The term "retrospective" stresses that inference is made *after* data is observed. In real time the prior is used for forecasting future variables while learning can improve only *future* forecasts. Under the simplification that there is only one shock $\hat{\zeta}_t$, agents believe the true Markov transitions are (see (18a) with $\lambda_{\zeta}^s=1$) $\hat{\zeta}_{t+1} - \lambda_{\zeta}\hat{\zeta}_t = s_t + \varepsilon_{t+1}^{\zeta}$, $\varepsilon_{t+1}^{\zeta} \sim N(0, 1/\nu)$. From the data they discover λ_{ζ} and we assume they are sure what ν is but not what the "regimes" parameters s_t are. This is a strong assumption but uncertainty about both(ν, s_t) raises technical difficulties. The infinite number of parameters s_t expresses the non-stationary economy. They reflect changed technologies and social institutions and since commodities change over time, s_t actually represent different objects and a single commodity over time is a simplification.

We now suggest that the structure of changing parameters requires us to supplement the standard Bayesian inference. To explain why note that at $t-1$ an agent has a prior about s_{t-1} used to forecast $\hat{\zeta}_t$. After observing $\hat{\zeta}_t$ he updates the prior into a sharper posterior estimate $E_t^i(s_{t-1}|\hat{\zeta}_t)$ of s_{t-1} which, as a random variable, we denote by $s_{t-1}(\hat{\zeta}_t)$. But at date t he needs to forecast $\hat{\zeta}_{t+1}$. For that he does not need a posterior estimate of s_{t-1} but rather, *a new prior on* s_t! Agents do not know if and when parameters change. If they knew s_t changes slowly or $s_t = s_{t-1}$ then an updated posterior of s_{t-1} is a good prior of s_t. Without knowledge, they presume $s_{t-1} \neq s_t$ is possible and seek additional information to arrive at a sharper subjective estimate of s_t. Public *qualitative* information is an important source which offers a route to such alternative estimate.

4.2.1. *Qualitative information as a public signal*

Quantitative data like $\hat{\zeta}_t$ arrive with *qualitative* information about unusual conditions under which the data was generated. For example, if $\hat{\zeta}_t$ are profits of a firm then $\hat{\zeta}_t$ is a number in a financial report which contains *qualitative* information about changing consumer taste, new products, technology, joint ventures, research & development etc. If $\hat{\zeta}_t$ reflect measures of productivity then a great deal of *qualitative* information is available about technologic discoveries, new products or new processes. If $\hat{\zeta}_t$ is growth rate of GDP much public information is available about business conditions, public policy or political environment. Qualitative information cannot, in general, be compared over time and does not constitute conventional "data". To avoid complex modeling, we simply translate Kurz's (2008) approach to qualitative information into

[4] Lack of merging of opinions follows from Rationality Principle 3 since any Bayesian updating procedure with infinite number of unknown parameters does not converge, even if the data is i.i.d. For a formal treatment see Freedman (1963, 1965). The main effect of Rationality Principle 3 is the specific Markov distribution of the limiting Bayes estimates.

[5] The role of belief dynamics is essential in this paper and its foundations are presented in Section 4.2. However, this section is technical in nature and a first time reader who takes (21) as given can maintain continuity of the paper's development by skipping to Section 4.3 and returning to Section 4.2 after completing the explorations of monetary policy.

date t qualitative *public signal* which allows an agent to form a subjective belief about s_t. Since it is based on qualitative information it is naturally open to diverse subjective assessments. More specifically, we assume at date t, in addition to data $\hat{\zeta}_t$, there is a public signal leading agent i to formulate an alternate prior on s_t which, as a random variable, we denote by S_t^i defined by

$$S_t^i \sim N\left(\Psi_t^i, \frac{1}{\gamma}\right)$$

We interpret Ψ_t^i as a prior subjective mean deduced from the public signal. One can say either that i "observes" Ψ_t^i and γ or that he assesses these values from a qualitative public signal and public data. The main question is how to reconcile S_t^i with the posterior $\mathbf{s}_{t-1}(\hat{\zeta}_t)$ formulated earlier, given the data $\hat{\zeta}_t$. To do that we specify the updating process.

4.2.2. A Bayesian inference: beliefs are Markov state variables with transition (21)

Agents believe (18a) with $\lambda_\zeta^s = 1$ is the truth with known precision ν. At $t-1$ (say $t-1=-1$) he forecasts $\hat{\zeta}_t$ and uses a prior about s_{t-1}described by $s_{t-1}^i \sim N(s, \frac{1}{\alpha})$. At t (here $t=0$), after observing s_t (recall $\varepsilon_{t+1}^\zeta \sim N(0,(1/\nu))$) the posterior on $s_{t\text{-}1}$is updated to be

$$E_t^i(s_{t-1}|\hat{\zeta}_t) = \frac{\alpha s + \nu[\hat{\zeta}_t - \lambda_\zeta \hat{\zeta}_{t-1}]}{\alpha + \nu}, \quad s_{t-1}(\hat{\zeta}_t) \sim N[E_t^i(s_{t-1}|\hat{\zeta}_t), \frac{1}{\alpha+\nu}]$$

Using the qualitative public signal, agent i makes the assessment $S_t^i \sim N(\Psi_t^i,(1/\gamma))$ independently of the random variable $s_{t-1}(\hat{\zeta}_t)$ and we have two alternative priors. The assumption made is as follows.

Assumption 3. With subjective probability μ agent i forms date t prior belief about s_t defined by

$$s_t(\hat{\zeta}_t, \Psi_t^i) = \mu s_{t-1}(\hat{\zeta}_t) + (1-\mu) S_t^i, \quad 0 < \mu < 1$$

More generally, if at any stage $s_t(\hat{\zeta}_{t+1}, \Psi_t^i)$ is a posterior updated only by $\hat{\zeta}_{t+1}$, a revised prior given the subjective assessment S_{t+1}^i is defined by[6]

$$s_{t+1}(\hat{\zeta}_{t+1}, \Psi_{t+1}^i) = \mu s_t(\hat{\zeta}_{t+1}, \Psi_t^i) + (1-\mu) S_{t+1}^i,$$

$$E_{t+1}^i(s_{t+1}|\hat{\zeta}_{t+1}, \Psi_{t+1}^i) = \mu E_{t+1}^i(s_t|\hat{\zeta}_{t+1}, \Psi_t^i) + (1-\mu)\Psi_{t+1}^i \quad 0 < \mu < 1.$$

Theorem 1. *If Assumption 3 holds then for large t* $\Gamma(s_t) =$ Precision $(s_t|\hat{\zeta}_t, \Psi_t^i)$ converges to a constant Γ^* *but the Bayes estimate* $E_t^i(s_t|\hat{\zeta}_t, \Psi_t^i)$ *fluctuates indefinitely. Let the posterior belief of i about* s_t *be defined by* $g_t^i = E_t^i(s_t|\hat{\zeta}_t, \Psi_t^i)$. *Then this index is a Markov state variable and (21) holds.*

$$g_{t+1}^i = \lambda_Z g_t^i + \lambda_Z^\zeta[\hat{\zeta}_{t+1} - \lambda_\zeta \hat{\zeta}_t] + \rho_{t+1}^{ig}, \quad \rho_{t+1}^{ig} \sim N(0, \sigma_g^2)$$

with $\rho_{t+1}^{ig} = (1-\mu)\Psi_{t+1}^i$: *Assumption 3 implies (21).*

Proof. See Appendix A

The random component $\rho_{t+1}^{ig} = (1-\mu)\Psi_{t+1}^i$ arises from random arrival of commonly observed qualitative public signals subjectively interpreted by each agent. Correlation across agents is then a direct consequence of the fact that all observe the same public qualitative signal. Restrictions on the parameters $(\lambda_Z, \lambda_Z^\zeta)$ are explored in Appendix A and in Section 4.3.

4.3. Modeling diverse beliefs: market belief and the central role of correlation

The fact that individual beliefs fluctuate implies market belief (i.e. the distribution of g_t^i) may also fluctuate and uncertainty about an agent future belief imply that future market belief is also uncertain. Indeed, market belief is a crucial macroeconomic uncertainty which needs to be explored.

Averaging (21), denote by the mean of Z_t the cross sectional distribution of g_t^i and refer to it as "average market belief." It is observable. Due to correlation across agents' ρ_t^{ig}, the law of large numbers does not apply and the average of ρ_t^{ig} over i does not vanish. We write it in the form

$$Z_{t+1} = \lambda_Z Z_t + \lambda_Z^\zeta[\hat{\zeta}_{t+1} - \lambda_\zeta \hat{\zeta}_t] + \tilde{\rho}_{t+1}^Z \tag{22}$$

The distribution of $\tilde{\rho}_{t+1}^Z$ is unknown and may vary over time. But the fact that this random term is present reveals that the dynamics of Z_t depends upon the *correlation* across agents' beliefs. Had ρ_t^{ig} in (21) been independent across i, the law of large numbers would have implied $\tilde{\rho}_t^Z = 0$ hence the correlation ensures market belief does not degenerate into a relation $Z_{t+1} = \lambda_Z Z_t + \lambda_Z^\zeta[\hat{\zeta}_{t+1} - \lambda_\zeta \hat{\zeta}_t]$. Since correlation is not determined by individual rationality it becomes *an important belief*

[6] $E_t^i(s_t|\hat{\zeta}_t, \Psi_t^i)$ is the notation for date *t prior* belief about s_t used to forecast $\hat{\zeta}_{t+1}$. We then use $E_{t+1}^i(s_t|\hat{\zeta}_{t+1}, \Psi_t^i)$ for the posterior belief *about the same* s_t given the observation of $\hat{\zeta}_{t+1}$ but without changing the estimate of Ψ_t^i. Assumption 3 uses this posterior belief as a building block to construct the prior $E_{t+1}^i(s_{t+1}|\hat{\zeta}_{t+1}, \Psi_t^i)$ about the new parameter s_{t+1}.

externality. In sum, random individual belief translates into macro-uncertainty about future market belief. This uncertainty plays a central role in the theory and correlation externality is the basis for such uncertainty.

Since Z_t are observable, market participants have data on $\{(\hat{\zeta}_t, u_t, Z_t), t=1,2,\ldots\}$ and know the *joint empirical distribution* of these variables. We assume this distribution is Markov and to consider one exogenous variable at a time we have two alternative empirical distributions. The first corresponds to an economy with only technology shocks. It is described by the system with $u_t=0$ and empirical transitions

$$\hat{\zeta}_{t+1} = \lambda_\zeta \hat{\zeta}_t + \rho^\zeta_{t+1} \tag{23a}$$

$$Z_{t+1} = \lambda_Z Z_t + \lambda^\zeta_Z [\hat{\zeta}_{t+1} - \lambda_\zeta \hat{\zeta}_t] + \rho^Z_{t+1} \tag{23b}$$

$$\begin{pmatrix} \rho^\zeta_{t+1} \\ \rho^Z_{t+1} \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \sigma^2_\zeta, & 0 \\ 0, & \sigma^2_Z, \end{bmatrix} \right), \quad \text{i.i.d}$$

The second is associated with the two shocks $(\hat{\zeta}_t, \mathrm{u}_t)$ with a Markov empirical probability that has a transition function described by the system of equations of the form

$$\hat{\zeta}_{t+1} = \lambda_\zeta \hat{\zeta}_t + \rho^\zeta_{t+1} \tag{24a}$$

$$\mathrm{u}_{t+1} = \lambda_\mathrm{u} \mathrm{u}_t + \rho^\mathrm{u}_{t+1} \tag{24b}$$

$$\mathrm{Z}_{t+1} = \lambda_\mathrm{Z} \mathrm{Z}_t + \lambda^\zeta_\mathrm{Z} [\hat{\zeta}_{t+1} - \lambda_\zeta \hat{\zeta}_t] + \lambda^\mathrm{u}_\mathrm{Z} [\mathrm{u}_{t+1} - \lambda_\mathrm{u} \mathrm{u}_t] + \rho^\mathrm{Z}_{t+1} \tag{24c}$$

$$\begin{pmatrix} \rho^\zeta_{t+1} \\ \rho^\mathrm{u}_{t+1} \\ \rho^\mathrm{Z}_{t+1} \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0, \\ 0 \end{matrix} \begin{bmatrix} \sigma^2_\zeta, & 0, & 0 \\ 0, & \sigma^2_\mathrm{u}, & 0 \\ 0, & 0, & \sigma^2_\mathrm{Z} \end{bmatrix} \right), \quad i.i.d$$

This is a combination of technology and policy shocks.

An agent who does not believe (23a) and (23b) or (24a)–(24c) are the truth, formulates his own belief model. We describe an agent's perception of a two shocks model with state variables $(\hat{\zeta}^i_{t+1}, u^i_{t+1}, Z^i_{t+1}, g^i_{t+1})$ (see footnote 7). His belief takes the general form of a subjective perception model

$$\hat{\zeta}^i_{t+1} = \lambda_\zeta \hat{\zeta}_t + \lambda^g_\zeta g^i_t + \rho^{i\zeta}_{t+1} \tag{25a}$$

$$u^i_{t+1} = \lambda_u u_t + \lambda^g_u g^i_t + \rho^{iu}_{t+1} \tag{25b}$$

$$Z^i_{t+1} = \lambda_Z Z_t + \lambda^\zeta_Z [\hat{\zeta}_{t+1} - \lambda_\zeta \hat{\zeta}_t] + \lambda^u_Z [u_{t+1} - \lambda_u u_t] + \lambda^g_Z g^i_t + \rho^{iZ}_{t+1} \tag{25c}$$

$$g^i_{t+1} = \lambda_Z g^i_t + \lambda^\zeta_Z [\hat{\zeta}_{t+1} - \lambda_\zeta \hat{\zeta}_t] + \lambda^u_Z [u_{t+1} - \lambda_u u_t] + \rho^{ig}_{t+1} \tag{25d}$$

$$\begin{pmatrix} \rho^{i\zeta}_{t+1} \\ \rho^{iu}_{t+1} \\ \rho^{iZ}_{t+1} \\ \rho^{ig}_{t+1} \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}, \begin{bmatrix} \hat{\sigma}^2_\zeta, & 0, & 0, & 0 \\ 0, & \hat{\sigma}^2_u, & 0, & 0 \\ 0, & 0, & \hat{\sigma}^2_Z, & \hat{\sigma}_{Zg} \\ 0, & 0, & \hat{\sigma}_{Zg}, & \hat{\sigma}^2_g \end{bmatrix} \right)$$

Eqs. (25a)–(25d) show g^i_t pins down the transition of all state variables. This ensures one state variable pins down agent i's belief about how conditions at date $t+1$ are expected to be different from normal, where "normal" is represented by the empirical distribution. Comparing (24a)–(24c) with (25a)–(25d) shows that

$$\mathrm{E}^\mathrm{i}_\mathrm{t}[Z_{t+1}] = \lambda_Z Z_t + \lambda^\zeta_Z \lambda^g_\zeta g^i_t + \lambda^u_Z \lambda^g_u g^i_t + \lambda^g_Z g^i_t$$
$$\mathrm{E}^\mathrm{m}_\mathrm{t}[Z_{t+1}] = \lambda_Z Z_t$$

[7] Recall that the notation $(\hat{\zeta}^i_{t+1}, u^i_{t+1}, Z^i_{t+1})$ indicates agent i's *perception* of $(\hat{\zeta}_{t+1}, u^i_{t+1}, Z^i_{t+1})$. Since there is no difference between $\hat{E}^i_t, \hat{\zeta}^i_{t+1}$ and $\hat{E}^i_t, \hat{\zeta}_{t+1}$, we write $\hat{E}^i_t, \hat{\zeta}_{t+1}$ to express expectations of $\hat{\zeta}_{t+1}$ by i, *in accordance with his perception*.

hence

$$E_t^i\begin{pmatrix}\hat{\zeta}_{t+1}\\ u_{t+1}\\ Z_{t+1}\end{pmatrix}-E_t^m\begin{pmatrix}\hat{\zeta}_{t+1}\\ u_{t+1}\\ Z_{t+1}\end{pmatrix}=\begin{pmatrix}\lambda_\zeta^g\\ \lambda_u^g\\ \lambda_Z^\zeta\lambda_\zeta^g+\lambda_Z^u\lambda_u^g+\lambda_Z^g\end{pmatrix}g_t^i. \tag{26}$$

4.4. Some a-priori parameter restrictions

Rationality Principles 1–3 do not offer sufficient restrictions on the parameters of the perception models beyond those set by the data, such as λ_ζ. We later argue some restrictions are implied by the Bayesian learning procedure developed in Appendix A but the main restrictions are deduced from The Rational Belief principle (see Kurz, 1994) which restricts parameters of perception models by requiring the agent's belief, viewed as a dynamical system, to reproduce the empirical distribution which corresponds to these perception model. To illustrate consider the perception model in (25a)–(25d) relative to the empirical distribution in (24a)–(24c). It can be shown that, given the unconditional variance in (21), among the restrictions imposed by the Rational Belief principle are.

$$\begin{aligned}
&Var[\lambda_\zeta^g g_t^i+\rho_{t+1}^{i\zeta}]=Var[\rho_{t+1}^\zeta] \quad\Rightarrow\quad (\lambda_\zeta^g)^2\mathrm{Var}(g)+\hat{\sigma}_\zeta^2=\sigma_\zeta^2\\
&Var[\lambda_u^g g_t^i+\rho_{t+1}^{iu}]=Var[\rho_{t+1}^u] \quad\Rightarrow\quad (\lambda_u^g)^2\mathrm{Var}(g)+\hat{\sigma}_u^2=\sigma_u^2\\
&Var[\lambda_Z^g g_t^i+\rho_{t+1}^{iZ}]=Var[\rho_{t+1}^Z] \quad\Rightarrow\quad (\lambda_Z^g)^2\mathrm{Var}(g)+\hat{\sigma}_Z^2=\sigma_Z^2
\end{aligned} \tag{27}$$

Selecting a normalization we set $\lambda_\zeta^g=1$ hence the rationality conditions (27) imply

$$\mathrm{Var}(g)\le\sigma_\zeta^2,\ (\lambda_u^g)^2\mathrm{Var}(g)\le\sigma_u^2,\quad (\lambda_Z^g)^2\mathrm{Var}(g)\le\sigma_Z^2,\ \tilde{\sigma}_\zeta\le\sigma_\zeta,\ \tilde{\sigma}_u\le\sigma_u,\ \tilde{\sigma}_Z\le\sigma_Z \tag{28a}$$

In addition, it can be shown that the variance of ρ_{t+1}^z is restricted by σ_g^2 and is specified as

$$\sigma_Z^2\le\sigma_g^2 \text{ and has usually been set at } \sigma_Z=0.9\sigma_g. \tag{28b}$$

The unconditional variance of g_t^i can be calculated from (25d) as.

$$Var[g]=\frac{1}{(1-\lambda_Z^2)}\left[(\lambda_Z^\zeta)^2\sigma_\zeta^2+(\lambda_Z^u)^2\sigma_u^2+\sigma_g^2\right]. \tag{29a}$$

As can be seen from Appendix A the two parameters $(\lambda_Z^\zeta,\lambda_Z^u)$ arise from the learning feed-back. If we denote the variance which ignores such feed-back by $\mathrm{Var}^{NF}[g]$ then it would be defined by

$$\mathrm{Var}^{NF}[g]=\frac{\sigma_g^2}{(1-\lambda_Z^2)}. \tag{29b}$$

Clearly, Bayesian learning feed-back causes g_t^i to exhibit increased variance. Comparing the empirical distribution (24a) and (24b) with the perception model (25a) and (25b) shows that learning feed-back causes the belief variable g_t^i to introduce into (25a) and (25b) correlation with observed data which does not exist in the empirical distribution (24a) and (24b). For example, the empirical distribution shows that in the long run $Cov(\zeta_{t+1},\zeta_t)=\lambda_\zeta\,\mathrm{Var}(\zeta)$. This relation is not preserved in (25a) due to learning feed-back, as seen in (25d). Hence, on the face of it, a learning feed-back violates the Rational Belief principle. But a rational agent who learns in real time recognizes his perceived model exhibits higher variance than the empirical distribution and this fact raises two questions. First, how should we implement the rationality restrictions (28a) and (28b) to correct for the added variance due to learning feed-back? Second, what is a *reasonable* increased variance due to learning feed-back parameters $(\lambda_Z^\zeta,\lambda_Z^u)$ that should be permitted by a rational agent?

Our reply to the first question is to impose restrictions (28a) and (28b) subject to the estimated variance in (29b). Sufficient conditions to implement (28a) and (28b) are

$$\sigma_g=\sigma_\zeta\sqrt{(1-\lambda_Z^2)},\quad \lambda_u^g\le\frac{\sigma_u}{\sigma_\zeta},\quad \lambda_Z^g\le\frac{\sigma_Z}{\sigma_\zeta}\le0.9\sqrt{(1-\lambda_Z^2)} \tag{30}$$

Turning to the second question note that in most learning literature the increased variance is unrestricted and this may be taken as a form of bounded rationality. In the models of this paper this increased variance is expressed by the fact that if $\mathrm{Var}(g)$ is computed as in (29a) then

$$(\lambda_\zeta^g)^2\mathrm{Var}_g^2+\hat{\sigma}_\zeta^2>\sigma_\zeta^2,\quad (\lambda_u^g)^2\mathrm{Var}_g^2+\hat{\sigma}_u^2>\sigma_u^2,\quad (\lambda_Z^g)^2\mathrm{Var}_g^2+\hat{\sigma}_Z^2>\sigma_Z^2.$$

Appendix A shows we place a-priori restrictions on $(\lambda_Z^\zeta,\lambda_Z^u)$ by deducing them from the theory itself or from empirical evidence using forecast data. Appendix A shows $\lambda_Z^\zeta=(\mu\nu)/(\Gamma^*+\nu)$ where ν is the precision of the prior and Γ^* is limit precision of the posterior. Normally this ratio is relatively small, less than 0.25. The same applies to λ_Z^u. As to λ_z, the empirical evidence reveals (see Kurz and Motolese, 2011) high persistence of mean market belief and λ_z estimated in the range [0.6, 0.8]. With, $\lambda_z\simeq0.7$, $0\le\lambda_Z^\zeta\le0.25$, $0\le\lambda_Z^u\le0.25$, $0\le\lambda_Z^g\le0.64$, $\lambda_u^g=(\sigma_u/\sigma_\zeta)$ and conditions (30) used, all belief parameters are then set.

4.5. Definition of equilibrium

Having specified the belief of the agents, we define an equilibrium in the log-linearized economy.

Definition 2. Given a rule $\hat{r}_t = \xi_\pi \hat{\pi}_t + \xi_y \hat{y}_t + u_t$, an equilibrium in the log-linearized economy with two exogenous shocks is a stochastic process $\{(\hat{r}_t, \hat{p}_t, \hat{\pi}_t, \hat{w}_t, \hat{y}_t), t = 1,2,\ldots\}$ and a collection of decision functions $(\hat{c}_t^j, \hat{b}_t^j, \hat{\ell}_t^j, \hat{n}_t^j, \hat{y}_t^j, \hat{q}_{jt}^*)$ such that:

(i) decision functions are optimal for all j given j's belief (25a)–(25d),
(ii) markets clear: $\int_0^1 \hat{c}_t^j dj = \hat{y}_t$, $\int_0^1 \hat{b}_t^j\, dj = 0$, and $\int_0^1 \hat{\ell}_t^j dj = \int_0^1 \hat{n}_t^j\, dj$, and
(iii) j's borrowing is bounded, transversality conditions satisfied and equilibrium is determinate.

Optimal decision functions $(\hat{c}_t^j, \hat{b}_t^j, \hat{\ell}_t^j, \hat{n}_t^j, \hat{y}_t^j, \hat{q}_{jt}^*)$ are linear in state variables but the issues at hand are the *relevant* state variables. An equilibrium is said to be **regular** if it is expressed with finite state variables, a finite number of lagged endogenous variables and hence it is of finite memory. If x_t is a finite vector of state variables in a regular equilibrium of the log linearized economy then, *as an equilibrium condition*, an endogenous variable w_t has a reduced form $A_w x_t$ where A_w is a vector of parameters. An equilibrium is **irregular** if it is not regular. In such equilibria endogenous variables depend on an infinite number of lagged variables or on expectation over infinite number of forward looking variables which cannot be reduced to a finite set of past or present variables. Irregular equilibria are important but analytically more difficult to simulate. This is of particular importance when we vary the monetary rule and consider later other rules which are different from $\hat{r}_t = \xi_\pi \hat{\pi}_t + \xi_y \hat{y}_t + u_t$.

One uses standard dynamic programming to show that for the economy at hand equilibria leading to (16a)–(16c) with beliefs (25a)–(25d) are **regular**. Individual decisions are functions of the state variables $(Z_t, \hat{\zeta}_t, u_t, \hat{b}_{t-1}^j, g_t^j)$ while the equilibrium map of the macro-variables $(\hat{r}_t, \hat{p}_t, \hat{\pi}_t, \hat{w}_t, \hat{y}_t)$ is stated, as a set of functions of the state variables $(Z_t, \hat{\zeta}_t, u_t)$. The difference between state spaces relevant to *each individual agent* and state spaces relevant *to the macro economy* is an important outcome of individual belief diversity.

Comment 4.1: comparison with the Arrow–Debreu's view of risk

Equilibrium endogenous variables depend upon $(Z_t, \hat{\zeta}_t, u_t)$ and this highlights the fact that our equilibrium theory goes far beyond the modeling of risk in an Arrow–Debreu economy. This fact was first explored in Kurz (1994), Kurz and Wu (1996) and Kurz (1997) who define the added uncertainty in our model as "Endogenous Uncertainty." The essential point is that in an Arrow–Debreu economy the state space is defined only with respect to exogenous shocks while in our theory a central source of market risk is the future belief of agents. Since endogenous variables are functions of market belief and since agents need to forecast endogenous variables, they must forecast uncertain future market belief which, for each agent, is "the belief of others." In the log linear economy the belief of others is defined by Z_t and compared with an Arrow–Debreu economy, our state space is expanded from $(\hat{\zeta}_t, u_t)$ to $(Z_t, \hat{\zeta}_t, u_t)$. Since all three state variables are observed, there is no problem of infinite regress and higher order beliefs. Market belief is observed and is thus common knowledge. All agents form belief about future market belief in the way they form belief about any aggregate variable. Markets, however, are incomplete since data on belief (and on other variables such as unemployment) can only be collected by survey methods, making the data subject to errors in observation. Although such data is sufficient for economic analysis and decisions, it is hard to enforce contracts which are contingent upon information containing errors in measurement, a reason that may cause such contracts to be contested in court.

The expanded state space of our economy permits phenomena which an Arrow–Debreu economy does not permit. These are equilibrium movements which are volatile and "unexplainable" in the familiar way of pointing to exogenous shocks that "explain" events. Sudden shifts in belief distributions result in crashes of asset prices or dramatic moves of commodity prices. No technological or exogenous event can explain the 2007–2008 Financial Crisis but we do know that real estate and private housing portfolios were based on widely shared and correlated forecasts of ever rising real estate prices. When agents began to realize late in 2006 that these forecasts were highly correlated *mistaken* forecasts they first tried to unwind their leveraged positions. However, as expectations about future market belief shifted, asset prices began their long decline culminating in the financial crisis. These are exactly the phenomena captured by an equilibrium with endogenous uncertainty. No Black Swans or herding behavior with private information is needed to explain such events. All we need is for individual agents to hold diverse but correlated beliefs and the conditions required for our theory to hold are simple: that agents hold common information, that this fact is common knowledge, that our economic environment is a complex, changing and non-stationary, and that in a changing environment agents can never learn the true economic structure generating the data. This collective ignorance is then common knowledge. Even when subjected to Rationality Axioms, room is left for diverse models of deduction from available data, subjective but correlated interpretation of public signals available to all (in the form of qualitative information) and persistence of diverse but correlated posterior estimates and forecasts. A central implication of our theory insists that holding a Rational belief and being wrong are not contradictory, and a financial crisis can be ignited exactly by a large number of agents who commit substantial resources based on wrong and correlated forecasts. Indeed, the Rationality Axioms are exactly where one finds the formal route to explain the independent nature and persistence of the dynamics of belief, which is at the heart of the impact of expectations on market fluctuations.

We finally note this paper shows that, under the postulated beliefs, aggregation of equilibrium quantities is possible in the log linearized economy. The paper also studies the impact of diverse beliefs on the performance of this log-linearized economy. However, it is important to clarify the relationship between the microeconomic equilibrium of the log linearized economy and the macroeconomic model implied by it. To understand why, recall that in a representative agent economy a macroeconomic model is a solution of a single agent's dynamic optimization. With diverse beliefs this is not true. We explain below that to define the macroeconomic model one must solve the log-linearized micro-equilibrium from which to deduce key parameters needed for the macro-model. Hence, changes in policy require a reconstruction of the macro economy and to that end one must re-solve the micro-equilibrium. In short, equilibrium of the log-linearized micro-economy remains a basic tool needed for the functioning of the macro-model!

5. Equilibrium of the log-linearized economy and the Effects of diverse Beliefs

A macro-model requires a solution of the problems arising from the terms $(\Phi_t(\hat{c}),\Phi_t(\hat{q}))$ and the mean forecast operator $\overline{E}_t = \int_0^1 E_t^j\, dj$. The following provides a general answer to these questions.

Theorem 2. In an equilibrium of the log linearized economy with the policy rule $\hat{r}_t = \xi_\pi \hat{\pi}_t + \xi_y \hat{y}_t + u_t$:

(i) there exist parameters $(\lambda_c^\Phi,\lambda_q^\Phi)$ such that $\Phi_t(\hat{c}) = \lambda_c^\Phi Z_t$ and $\Phi_t(\hat{q}) = \lambda_q^\Phi Z_t$,
(ii) there exist parameters (Γ^y,Γ^π) such that

$$\overline{E}_t(\hat{y}_{t+1}) = E_t^m(\hat{y}_{t+1}) + \Gamma^y Z_t$$
$$\overline{E}_t(\hat{\pi}_{t+1}) = E_t^m(\hat{\pi}_{t+1}) + \Gamma^\pi Z_t$$

Theorem 2 formulates transformations which are parts of the equilibrium conditions and these do not hold with respect to non-linear functions of macro-variables. The proof of Theorem 2 will clarify this last comment. However, the main reason for including the proof here is that it introduces concepts and notation used in the rest of this paper. Hence the reader needs a minimal familiarity with them.

Proof: To explain the parameters $(\lambda_c^\Phi,\lambda_q^\Phi,\Gamma^y,\Gamma^\pi)$ we outline the key points of a proof of Theorem 2 for the case of two exogenous variables. Solutions of endogenous variables in the log linear economy are linear in state variables. Keeping in mind (25a)–(25d), individual decision functions have the general form.

$$\hat{c}_t^j = A_y^Z Z_t + A_y^\zeta \hat{\zeta}_t + A_y^u u_t + A_y^b \hat{b}_{t\text{-}1}^j + A_y^g g_t^j \equiv A_y \bullet \left(Z_t,\hat{\zeta}_t,u_t,\hat{b}_{t-1}^j,g_t^j\right) \tag{31a}$$

$$\hat{q}_{jt}^* = \frac{\omega}{1-\omega}\left[A_\pi^Z Z_t + A_\pi^\zeta \hat{\zeta}_t + A_\pi^u u_t + A_\pi^b \hat{b}_{t-1}^j + A_\pi^g g_t^j\right] \equiv \frac{\omega}{1-\omega} A_\pi \bullet \left(Z_t,\hat{\zeta}_t,u_t,\hat{b}_{t-1}^j,g_t^j\right) \tag{31b}$$

$$\hat{b}_t^j = A_b^Z Z_t + A_b^\zeta \hat{\zeta}_t + A_b^u u_t + A_b^b \hat{b}_{t\text{-}1}^j + A_b^g g_t^j \equiv A_b \bullet \left(Z_t,\hat{\zeta}_t,u_t,\hat{b}_{t-1}^j,g_t^j\right) \tag{31c}$$

Equilibrium conditions, $\int_0^1 \hat{c}_t^j dj = \hat{y}_t$, $\int_0^1 \hat{b}_t^j\, dj = 0, Z_t = \int_0^1 g_t^j\, dj$ and $\int_0^1 \hat{q}_{jt}^* = (\omega/(1-\omega))\hat{\pi}_t$ imply the aggregates

$$\hat{y}_t = A_y^Z Z_t + A_y^\zeta \hat{\zeta}_t + A_y^u u_t + A_y^b 0 + A_y^g Z_t \equiv A_y \bullet (Z_t,\hat{\zeta}_t,u_t,0,Z_t) \tag{31d}$$

$$\hat{\pi}_t = A_\pi^Z Z_t + A_\pi^\zeta \hat{\zeta}_t + A_\pi^u u_t + A_\pi^b 0 + A_\pi^g Z_t \equiv A_\pi \bullet (Z_t,\hat{\zeta}_t,u_t,0,Z_t) \tag{31e}$$

$$\hat{q}_t = \frac{\omega}{1-\omega}\left[A_\pi^Z Z_t + A_\pi^\zeta \hat{\zeta}_t + A_\pi^u u_t + A_\pi^b 0 + A_\pi^g Z_t\right] \equiv \frac{\omega}{1-\omega} A_\pi \bullet (Z_t,\hat{\zeta}_t,u_t,0,Z_t) \tag{31f}$$

To compute $\Phi_t(\hat{c})$ note $\hat{c}_t = \hat{y}_t$ and use (31a)–(31f), (25a)–(25d) to deduce that

$$\int_0^1 (E_t^j(\hat{c}_{t+1}^j) - E_t^j(\hat{c}_{t+1}))dj = A_y^g\left[\int_0^1 \left(E_t^j(g_{t+1}^j) - E_t^j(Z_{t+1})\right)dj\right] = -A_y^g \lambda_Z^g Z_t$$

hence

$$\lambda_c^\Phi = -A_y^g \lambda_Z^g,$$
$$\int_0^1 (E_t^j \hat{q}_{j(t+1)} - E_t^j \hat{q}_{(t+1)})dj = -\frac{\omega}{1-\omega} A_\pi^g \lambda_Z^g Z_t$$

hence

$$\lambda_q^\Phi = -(\omega/(1-\omega)) A_\pi^g \lambda_Z^g$$

Using the same information and (26), compute now the expression

$$\int_0^1 [E_t^j \hat{y}_{t+1} - E_t^m \hat{y}_{t+1}] = (A_y^Z + A_y^g)\int_0^1 [E_t^j Z_{t+1} - E_t^m Z_{t+1}] + A_y^\zeta \int_0^1 [E_t^j \hat{\zeta}_{t+1} - E_t^m \hat{\zeta}_{t+1}] + A_y^u \int_0^1 [E_t^j u_{t+1} - E_t^m u_{t+1}]$$

$$=((A_y^Z+A_y^g)[\lambda_Z^\zeta\lambda_\zeta^g+\lambda_Z^u\lambda_u^g+\lambda_Z^g]+A_y^\zeta\lambda_\zeta^g+A_y^\zeta\lambda_u^g)Z_t$$

Similar argument holds with respect to inflation hence we have that

$$\Gamma^y=(A_y^Z+A_y^g)[\lambda_Z^\zeta\lambda_\zeta^g+\lambda_Z^u\lambda_u^g+\lambda_Z^g]+A_y^\zeta\lambda_\zeta^g+A_y^u\lambda_u^g \tag{31g}$$

$$\Gamma^\pi=(A_\pi^Z+A_\pi^g)[\lambda_Z^\zeta\lambda_\zeta^g+\lambda_Z^u\lambda_u^g+\lambda_Z^g]+A_\pi^\zeta\lambda_\zeta^g+A_\pi^u\lambda_u^g \tag{31h}$$

Note these transformations do not hold for, say $(\hat{y}_t)^2$ which is not a linear function of state variables. □

To study the system we transform it into one in which the expectation operator obeys the law of iterated expectations so as to enable us to use standard techniques of analysis. To do that we observe that for defining the macro-model one needs only the two parameters defined by

$$B_y=\lambda_c^\Phi+\Gamma^y+\left(\frac{1}{\sigma}\right)\Gamma^\pi,\quad B_\pi=(1-\omega)\lambda_q^\Phi+\Gamma^\pi.$$

Using the theorem above we can now rewrite system (16a)–(16c) in the form

$$\text{IS curve}\quad \hat{y}_t=E_t^m(\hat{y}_{t+1})+B_yZ_t-\left(\frac{1}{\sigma}\right)[r_t-E_t^m(\hat{\pi}_{t+1})] \tag{32a}$$

$$\text{Phillips curve}\quad \hat{\pi}_t=\kappa(\eta+\sigma)\hat{y}_t+\beta E_t^m\hat{\pi}_{t+1}+\beta B_\pi Z_t-\kappa(1+\eta)\hat{\zeta}_t \tag{32b}$$

$$\text{Monetary rule}\quad \hat{r}_t=\xi_\pi\hat{\pi}_t+\xi_y\hat{y}_t+u_t \tag{32c}$$

together with the law of motion of $(\hat{\zeta}_t,u_t,Z_t)$ under the empirical transitions (24a)–(24c). Since this system is operative under a single probability law m which satisfies the law of iterated expectations, standard methods of Blanchard–Kahn (1980) are applicable for setting conditions to ensure determinacy.

The system at hand shows that diverse beliefs have two effects. First, the mean market belief Z_t has an amplification effect on the dynamics of the economy. The second is more subtle. To explain it note the probability in (32a) and (32b) is m, not the true dynamics (18a) and (18b) that is unknown to anyone and simulations are conducted with respect to the empirical probability m. Hence, (32a)–(32c) may not reflect changes in s_t (see (18a) and (18b)) if they are not predicted by the public and expressed in Z_t. But this fact shows that a central bank faces an enduring problem for which only imperfect solutions exist. To capture what it does not observe, the bank has two options. One is to base policy upon market belief, expressed either directly by Z_t or by asset prices which are functions of Z_t (see Kurz and Motolese, 2011). A second option is to use the bank's own belief model in making policy decisions, giving rise to what the market would view as random policy shocks, which may turn out to be costly in becoming an independent cause of volatility. We return to this subject later when we discuss the implications to monetary policy.

5.1. *Some characteristics of the microeconomic equilibrium*

It follows from (31g) and (31h) that (B_y, B_π) are functions of (A_y, A_π, A_b) which is an equilibrium of the log-linearized micro-economy. That is, to deduce a solution of the macro-model (32a)–(32c), one must first obtain a *micro-equilibrium* solution of (A_y, A_π, A_b). Note that an equilibrium depends upon the model parameters including policy parameters. Since we study the impact of changes in policy parameters, the shape of the map from parameters to equilibria (A_y, A_π, A_b) is important. For this reason we use the term "Equilibrium Manifold" to describe the set of equilibria (A_y, A_π, A_b) as a function of the model's parameters.

Appendix B reviews computation of (A_y, A_π, A_b) for a simple model of a technology shock with $u_t=0$. Exploring this model further, note it is a system with endogenous variables $(\hat{\pi}_t,\hat{y}_t)$ and shocks $(\hat{\zeta}_t,Z_t)$. Rewriting the system in a standard manner we have

$$\hat{\zeta}_{t+1}=\lambda_\zeta\hat{\zeta}_t+\rho_{t+1}^\zeta \tag{33a}$$

$$Z_{t+1}=\lambda_ZZ_t+\lambda_Z^\zeta[\hat{\zeta}_{t+1}-\lambda_\zeta\hat{\zeta}_t]+\rho_{t+1}^Z \tag{33b}$$

$$\begin{pmatrix}\rho_{t+1}^\zeta\\ \rho_{t+1}^Z\end{pmatrix}\sim N\left(\begin{matrix}0\\0\end{matrix},\begin{bmatrix}\sigma_\zeta^2,\sigma_{\zeta Z}\\ \sigma_{\zeta Z},\sigma_Z^2,\end{bmatrix}\right)$$

$$E_t^m(\hat{y}_{t+1})=\hat{y}_t+\left(\frac{1}{\sigma}\right)[\xi_\pi\hat{\pi}_t+\xi_y\hat{y}_t-E_t^m(\hat{\pi}_{t+1})]-B_yZ_t,\quad B_y=\lambda_c^\Phi+\Gamma^y+\left(\frac{1}{\sigma}\right)\Gamma^\pi \tag{33c}$$

$$E_t^m\hat{\pi}_{t+1}=\frac{1}{\beta}\hat{\pi}_t-\frac{\kappa(\eta+\sigma)}{\beta}\hat{y}_t-B_\pi Z_t-\frac{1}{\beta}\kappa(1+\eta)\hat{\zeta}_t,\quad B_\pi=(1-\omega)\lambda_q^\Phi+\Gamma^\pi,\quad \kappa=\frac{(1-\beta\omega)(1-\omega)}{\omega} \tag{33d}$$

Eqs. (32a)–(32c) together with (33a)–(33d) show that endogenous variables do not affect the dynamics of either the exogenous shock or the dynamics of belief. It then follows that we have Proposition 3.

Proposition 3. *Determinacy of equilibrium is not affected by diversity of beliefs.*

Proof. It follows from Blanchard–Kahn (1980) that to compute the relevant eigenvalues one ignores the first two equations. For the case $u_t = 0$, the condition for determinacy when $\xi_y \geq 0$, $\xi_\pi \geq 0$ is

$$\xi_y(1-\beta) + (\xi_\pi - 1)\kappa(\eta+\sigma) > 0. \tag{34}$$

It does not involve belief parameters and is the same as an equivalent model with homogenous beliefs. □

Does Proposition 3 mean that existence and uniqueness are the same as they would be without diverse beliefs? The answer is No. To explain why, we continue to study the simple version of the model with a single exogenous technology shock. We now explore some features of the equilibrium system.

Proposition 4. *It is impossible to solve the macro-model using only the Macro-system (33a)–(33d). To solve (33a)–(33d) one must first deduce (B_y, B_π) from a micro-equilibrium of the log-linearized economy underlying (33a)–(33d).*

Proof. See Appendix A □

Comment 5.1: interpretation of equilibrium; have we just created a new representative agent?

This is an appropriate time to clarify what Theorem 2 says about the meaning of aggregation. After all, our starting point is a diverse agent economy which we reduced to an economy where the aggregates follow their equilibrium path based on one probability measure, independently of the diversity with which we started. If diversity plays no role in the equilibrium path, have we not created a new representative agent? We offer two answers. We first note that our methods and assumptions actually remove some effects of diversity but second, we explain why diversity is still at the heart of our economy.

By design, an equilibrium of a log linear economy disregards variances, focusing on mean values. In addition, the sampling Assumption 1 and Insurance Assumption 2 are designed to disregard income and distributional effects. It is thus no surprise that out of the diversity of beliefs represented by the distribution of g_t^i, all we find in the equilibrium of Theorem 2 is only the mean Z_t. Hence, our assumptions ensure that for any policy parameters, the dynamics of the aggregates in the linearized economy do not depend upon time changes in the distribution of the heterogenous economy underneath the aggregates. This conclusion is in conflict with the empirical evidence of Kurz and Motolese (2011) who show that asset prices and excess returns change in response to changes in the cross-sectional distribution of the g_t^i. It is thus clear that our model provides a first approximation in which some distributional effects on the aggregates have been assumed away. But then, if one is to argue that Theorem 2 does not lead to a new representative agent economy, what is the role of belief diversity in the functioning of the model?

Our answer to the last question has two parts. One is formal and the second is a positive explanation of the role of diversity in the model's functioning. As to the formal part, since Z_t is the mean belief, individual agents define their own beliefs relative to how they differ from Z_t who, for each of them, reflects the belief of "others." Hence, as a single agent economy the model in its present form loses its meaning. Under common knowledge of a single belief, agents know $Z_t = g_t^i$ hence they also know they do not need to forecast $Z_t \neq g_t^i$. Under such condition we must drop (22) and all transitions and perceptions like (23b), (24b) or (25c). With these changes the macro-system derived from Theorem 2 is not valid any longer. In short, without a background of belief diversity the model has no reasonable interpretation. *By itself*, Z_t reflects belief diversity. A complimentary argument insists that the distinction between g_t^i and Z_t results from the assumption of a non-stationary environment, which implies both the dynamics (21) of g_t^i and the belief diversity in the market. As long as we assume a non-stationary environment and deduce (21) we must also maintain a distinction between g_t^i and Z_t. The fact that we reduced, via linearization and Assumptions 1 and 2, the causal impact of diversity on the aggregates does not mean that this needs to be the case in all future versions of the model. One important extension was proposed by Kurz and Motolese (2001) who show (p. 533) that changes in the cross-sectional distribution of the g_t^i changes the variance of Z_t. To implement this idea in our model recall that since the g_t^i in (21) are correlated, the random variable $\tilde{\rho}_{t+1}^Z$ in (22) is the result of averaging (21), leading to the definition

$$lim_{K\to\infty} \frac{1}{K}\sum_{i=1}^{K} \rho_{t+1}^{ig} = \tilde{\rho}_{t+1}^Z.$$

The correlation across the ρ_{t+1}^{ig} is a belief externality, not subject to the rationality of any agent. Hence, it can exhibit complex behavior and the empirical distribution of $\tilde{\rho}_{t+1}^Z$ does not need to have a constant variance σ_Z^2, as assumed in (23b) above. Alternatively, $\tilde{\rho}_{t+1}^Z$ can exhibit stochastic volatility, allowing the variance σ_Z^2 to be a function of the cross sectional variance of the ρ_{t+1}^{ig}. There is substantial evidence for stochastic volatility in asset returns and in other macro economic variables related to financial markets. The introduction of such stochastic volatility into our equilibrium would be a novelty of its own.

Turning now to the positive explanation we suggest that our model is intended first of all to study the interaction between policy and market belief hence belief diversity plays an important role in two pivotal aspects of the model. First, diversity has an effect on the aggregates via its interaction with policy. Proposition 4 shows that in order to derive the solution of the aggregates one must first compute the parameters (B_y, B_π), derived from the micro-equilibrium. Any change in policy changes the micro-equilibrium and hence (B_y, B_π). Second, since monetary policy is formulated as a function of the aggregates, the model aims to study how aggregates, via their impact on policy, affect the distribution and volatility of debt holdings and consumption. That is, instead of asking what is the effect of the distributions on the aggregates we suggest that the reverse question is very important. Hence, we question what is the effect of policy (and the resulting aggregates) on the distribution and volatility of individual consumption and debt holdings. One of the important conclusions of this paper is that the current exclusive focus on the aggregates is probably an error of policy. Belief diversity causes individual consumption to be different from mean consumption which, in this model, equals output per capita. But if individual consumption is different from per capita output then welfare considerations suggest that policy should be concerned with individual consumption volatility at least as much as it is interested in the stability of the aggregates. In fact, policies that stabilize the aggregates often cause high volatility of interest rates and financial markets, leading to volatile bond holding and increased individual consumption volatility. In short, the fact that the economy is heterogenous remains a central fact that has an impact on the way we use the model for policy analysis and on the way we interpret its results. We now return to exploring the characteristics of the micro-equilibrium.

Proposition 4 raises questions of existence and uniqueness of equilibria (A_y, A_π, A_b) in the log linearized economy. The system of equations in the proof of Proposition 4 (Appendix A) implied by parameter matching is non-linear due to the presence of *products* which arise from the borrowing function (Appendix A, Eqs. (A.8a)–(A.8c)). Indeed, there are 8 such products: $(A_y^b A_b^z, A_y^b A_b^\zeta, A_y^b A_b^b, A_y^b A_b^g)$ and $(A_\pi^b A_b^Z, A_\pi^b A_b^\zeta, A_\pi^b A_b^b, A_\pi^b A_b^g)$. A close inspection leads to the following observation.

Proposition 5. *The equation system defining equilibrium for $\boldsymbol{\tau}_b > 0$ is non-linear with at least two solutions where one entails explosive optimal borrowing (i.e. Ponzi), independent of determinacy conditions.*

Proof. See Appendix A. □

The dynamic determinacy condition (34) plays no role in Proposition 5. This fact suggests that determinacy conditions restrict the map which defines the micro-economic equilibrium only partly. This map can exhibit properties with important consequences (e.g. for individual borrowing patterns) which are not implied by dynamic determinacy. This, indeed, is the conclusion which is now explored.

5.2. *Understanding equilibrium consequences of interactions of policy and beliefs*

Selecting a non-explosive solution of A_y^b and $A_\pi^b = 0$ proved in Proposition 5 reduces the system to six linear equations in six unknowns (A_y, A_π). Denote this equation system by MA=f and inspection of (A.8a) and (A.8b) in Appendix A reveal that terms included in the right hand vector *f are only parameters of the exogenous shocks* to the system. Hence, altering these shocks impacts the equilibrium solution of *A*. But also, for a system without shocks $f=0$, and there is no micro-economic equilibrium since belief variables are redundant: there is no equilibrium if one cannot define what beliefs are about. We refer to the determinant $|M|$ as the "*Equilibrium Determinant.*" It helps understand the impact of diverse beliefs on policy since changes in policy parameters change the values of (A_y, A_π, A_b) and (B_y, B_π) in (33a)–(33d), and as we sweep over the feasible space of policy parameters (ξ_y, ξ_π) the determinant changes. We know the basic fact that

Proposition 6. *As one varies policy over the policy parameter space, the Equilibrium Determinant takes the value zero and changes sign.*

Note that Proposition 6 is not restricted to a determinacy region. As we sweep over feasible policies *that satisfy determinacy* (34), we sweep over the Equilibrium Manifold and raise two questions which are pivotal for the impact of policy on (σ_y, σ_π):

(i) Do (ξ_y, ξ_π) have a monotonic effect on (σ_y, σ_π) and what is the implied trade-off between them?
(ii) Do (ξ_y, ξ_π) have a continuous effect on (σ_y, σ_π)?

To resolve the question of continuity we later provide a simulation example to show the Equilibrium Determinant of an economy with diverse beliefs may take zero value and change sign inside the region of determinacy. This is not true for rational expectations under which determinacy excludes a zero determinant. Hence, the equilibrium map of an economy with diverse beliefs may have singularity inside the determinacy region when (B_y, B_π) are unbounded and, by implication, in such economy (σ_y, σ_π) may not be continuous with respect to (ξ_y, ξ_π) within the region of determinacy. We now turn to the question of monotonicity.

To highlight the issues consider standard results of the RBC model (33a)–(33d) with $u_t = 0$ and RE under which all believe (24a) is the truth. Table 1.1 reports simulation results for $(\xi_y \geq 0, \xi_\pi \geq 1)$ which is the standard range used in most literature.[8]

[8] Our model is quarterly but we report annualized statistics in all tables hence the inflation rate in the model is one quarter the rate reported in the tables. For consistency, the policy weight on output is also annualized in the table and hence in the simulations the rule would be defined as $\hat{r}_t = \xi_\pi \hat{\pi}_t + (\xi_y/4)\hat{y}_t + u_t$.

Table 1.1
Annualized RBC Volatility and monotonicity under rational expectations. Standard RBC with $\sigma_\zeta = 0.0072$.

σ_y		ξ_y					σ_π		ξ_y				
		0.0	0.6	1.2	1.5	30			0.0	0.6	1.2	1.5	30
ξ_π	1.1	1.52	1.25	1.06	0.98	0.13	ξ_π	1.1	2.74	5.98	8.24	9.12	19.25
	1.6	1.68	1.57	1.47	1.43	0.38		1.6	0.86	2.15	3.28	3.80	16.30
	2.2	1.71	1.65	1.59	1.56	0.59		2.2	0.47	1.22	1.91	2.24	13.77
	2.8	1.72	1.68	1.64	1.62	0.75		2.8	0.33	0.85	1.34	1.58	11.91
	30	1.75	1.75	1.74	1.74	1.61		30	0.02	0.06	0.09	0.11	1.68

The specified quarterly model parameters are standard (e.g. Galí, 2008; Woodford, 2003), and will be maintained throughout:

$$\beta = 0.99, \quad \sigma = 0.9, \quad \eta = 1.0, \quad \tau_b = 10^{-4}, \quad \omega = \tfrac{2}{3}, \quad \theta = 6, \quad \lambda_\zeta = 0.90$$

Assuming $\beta = 0.99$ means the riskless annualized rate is 4% which is compatible with the empirical record although we do not aim here at exact calibration. The standard RBC assumption $\sigma_\zeta = 0.0072$ of (measured for the “Solow Residuals”) is being used and we comment on this matter below. Also, recall that the empirical record for the US exhibits $\sigma_y = 1.81$, $\sigma_\pi = 1.79$.

Table 1.1 shows that for $\sigma_\zeta = 0.0072$ there are policy configurations for which the simulated values are within range of the data. More important is the effect of policy. The conclusions are as follows:

- Increasing ξ_y results in a *monotonic* decrease of σ_y and a monotonic increase of σ_π.
- Increasing ξ_π results in a *monotonic* increase of $\sigma_y$and a monotonic decrease of σ_π.

These results imply a policy trade-off between σ_y and σ_π, offering the central bank a choice between these volatility measures. These results are consistent with the RE based model estimates of Taylor (1979), Fuhrer (1994), Ball (1999) and Rudebusch and Svensson (1999) that imply such a trade-off. Although the steps of parameter change are wide, choice of smaller steps shows the results are continuous with respect to policy.

Why study monotonicity properties at all? There are two reasons for that. First, monotonic responses render the impact of policy predictable since it means a central bank knows the *direction* of an effect of increasing weight of a policy instrument. In reality no central bank knows the exact effect of its instruments. Hence, a policy with non-monotonic effects means a bank is uncertain not only about the *size* of the effect of policy but even about the *direction* of effects. This is an undesirable state of affairs.

A second reason for interest in the monotonic effects of (ξ_y, ξ_π) or other policy parameters is the one studied by Taylor (1979), Fuhrer (1994), Ball (1999), Rudebusch and Svensson (1999) and others who explored the *policy trade-off* between σ_y and σ_π. Monotonic response is a precondition for the policy trade-off exhibited by these authors for their models. Non-monotonic response may imply that a central bank does not have a policy trade-off between σ_y and σ_π, a problem that would call for further examination.

Before proceeding we address the question of the size of the technological shock. The RBC approach assumed $\sigma_\zeta = 0.0072$. Strong objections were raised against this measure and persuasive case was made in support of the view that much of this residual is not technology (e.g. Basu, 1996; Eichenbaum, 1991; King and Rebelo, 1999; Summers, 1986). Most scholars suggest it should be no more than $\sigma_\zeta = 0.004$ hence we set $\sigma_\zeta = 0.004$. Table 1.2 shows the well known fact that under RE with a single technology shock and $\sigma_\zeta = 0.004$ the models' business cycles volatility virtually disappears (see King and Rebelo, 1999). We will later show that when diverse beliefs are introduced, volatility is amplified and fluctuations contain a major component due to market belief. Under such conditions the model exhibits realistic volatility and public stabilization policy of output and inflation becomes relevant. That is, central bank policy has an important objective of stabilizing the volatility amplification effect of market expectations!

6. An example: simulation results of the effects of diverse beliefs

6.1. *Some preliminary Issues: determinacy Conditions and output difference vs. output gap*

In this section we examine a two shock model with $u_t \neq 0$ and a wider policy space which allows $(\xi_y < 0, \xi_\pi \geq 1)$. This implies that we need to revise the determinacy conditions. A condition $(\xi_\pi < 1)$ clearly violates determinacy. It follows from Proposition 1 of Bullard and Mitra (2002) that for determinacy to hold when $\xi_y < 0$, condition (34) needs to be supplemented by the additional condition

$$\xi_y + \kappa(\eta + \sigma)\xi_\pi > -(1-\beta)\sigma. \tag{34a}$$

Table 1.2
Annualized RBC volatility and monotonicity under rational expectations. Standard RBC with $\sigma_\zeta = 0.004$.

σ_y		ξ_y					σ_π		ξ_y				
		0.0	0.6	1.2	1.5	30			0.0	0.6	1.2	1.5	30
ξ_π	1.1	0.84	0.69	0.59	0.55	0.07	ξ_π	1.1	1.52	3.32	4.58	5.07	10.69
	1.6	0.93	0.87	0.82	0.79	0.21		1.6	0.48	1.20	1.82	2.11	9.05
	2.2	0.95	0.92	0.88	0.87	0.33		2.2	0.26	0.68	1.06	1.24	7.65
	2.8	0.96	0.93	0.91	0.90	0.41		2.8	0.18	0.47	0.75	0.88	6.62
	30	0.97	0.97	0.97	0.95	0.89		30	0.01	0.03	0.05	0.06	0.93

Next we address the issue of output gap. Some NKMs under RE use output level under flexible prices as a yardstick for central bank policy. Under RE with flexible prices $\hat{y}_t^f$, the log deviation of output from steady state, is not a function of prices or expectations but only a function of the technology shock. Indeed, we can derive the relationship

$$\hat{y}_t^f = \left(\frac{1+\eta}{\sigma+\eta}\right)\hat{\zeta}_t.$$

Inserting this definition into the Phillips Curve (32b) transforms it into

$$\hat{\pi}_t = \kappa(\eta+\sigma)[\hat{y}_t - \hat{y}_t^f] + \beta E_t^m \hat{\pi}_{t+1} + \beta B_\pi Z_t \equiv \kappa(\eta+\sigma)\hat{x}_t + \beta E_t^m \hat{\pi}_{t+1} + \beta B_\pi Z_t \tag{35}$$

Where $\hat{x}_t = \hat{y}_t - \hat{y}_t^f$ The rest of (32a)–(32c) can then be redefined in terms of $\hat{x}_t$, including the monetary policy rule. It can be shown that this transformation is *equivalent* to solving the prior system in $(\hat{y}_t, \hat{\pi}_t)$ but altering the policy rule to be $\hat{r}_t = \xi_\pi \hat{\pi}_t + \xi_y[\hat{y}_t - ((1+\eta)/(\sigma+\eta))\hat{\zeta}_t]$ and no "output gap" needs be defined at all. The justification for this change in rule is that competitive equilibrium under flexible prices is the first best and hence policy should aim to attain it. This argument fails when we have diverse beliefs and/or other shocks such as a policy shock, since under flexible prices it is not an RBC model and $\hat{y}_t^f$ *is neither first best nor does it have any welfare significance*. We have noted earlier the results of Kurz et al. (2005) who show that diverse beliefs call, on their own, for stabilization policy that would counter the volatility amplification of market belief hence the policy objective should be $\hat{y}_t$ itself, which reflects the effect of belief, not $\hat{x}_t$. One can solve (35) by iterating forward and the solution of $\hat{\pi}_t$ is

$$\hat{\pi}_t = \sum_{\tau=0}^{\infty} \beta^\tau E_t^m[\kappa(\eta+\sigma)\hat{x}_{t+\tau} + \beta B_\pi Z_{t+\tau}] \tag{36}$$

showing that volatility of $\hat{x}_t$ is not a yardstick for policy. The case is made even stronger when other shocks, such as taste or cost push, are present and enter (36). Nevertheless, since in (36) $Z_{t+\tau}$ are not altered by policy, it is a purely mathematical observation without welfare implications that any policy that reduces the volatility of the gap $\hat{x}_t$ will necessarily also reduce the volatility of inflation $\hat{\pi}_t$.

Model simulations reveal belief diversity changes the conclusions in Table 1.1, particularly about monotonicity of response to policy and about policy trade-off. In the rest of this section we provide an example to explain these results. The example intends only to highlight the qualitative equilibrium results under diverse beliefs, rather than provide a precise calibration. Before doing so it is useful to give some intuition on how to think about an economy with diverse beliefs, why such an economy presents a complex challenge to central bank policy and why such complexity is absent from modeling with homogenous beliefs.

The starting point is a recognition that diverse expectations introduce into the market complex interactions. Expectations alter the motive of agents to consume, work, produce, borrow and invest in assets. While a technology shock increases present and future output, a change in expectations changes demand and output today. This is so since changed expectations entail a cascade of other effects such as expected higher wage rate in the future which can reduce the supply of labor today, raising wage rate and marginal cost today and these might lower output today but increase it in the future. Some effects of expectations are realized via borrowing. Keep in mind that although central bank policy acts on borrowing cost, agents' borrowing plays no role in the dynamics of a representative agent model since the agent does not borrow. This is not the case with diverse beliefs since in (33a)–(33d) market belief is key to the expectational complexity of the micro-equilibrium. The way agents act on their belief is borrowing and lending, and fluctuations in bond holdings are the key to fluctuations of individual consumption. Moreover, the volatility of individual consumption is different from the volatility of average consumption which, in equilibrium, equals per capita output. In short, changes in the distribution of beliefs have an impact on equilibrium wage rate, employment, borrowing and consumption– all effects not possible in a representative agent, RE based economy. Finally, policy also aims to change the motive for intertemporal allocation of consumption and labor! Hence, expectations may amplify, negate or distort the impact of policy but due to persistent diversity, central bank policy is always conducted with some opposition. These are all interaction effects which change the simple picture in Table 1.1. The visible effect of diverse beliefs is seen in the fact that when policy parameters change, the equilibrium map changes, altering the parameters (B_y, B_π) in (33a)–(33d) and hence (σ_y, σ_π).

6.2. The two shock (ζ, u) model (32a)–(32c) and individual consumption volatility

We allow $\xi_y<0$ as long as determinacy holds. Next, Table 2 provides results of simulating the diverse belief model (32a)–(32c) with realistic belief parameter values, most of which were motivated earlier: $\lambda_z=0.7, \lambda_u=0.7$, $\lambda_z^\zeta=0.25$, $\lambda_z^u=0.25$, $\lambda_z^g=0.2$, $\lambda_\zeta^g=1$, $\lambda_u^g=0.5$, $\sigma_\zeta=0.004$, $\sigma_u=0.002$, $\sigma_g=0.003$. The parameters $\lambda_u=0.7$, $\sigma_u=0.002$ were estimated using quarterly data on policy choices by the Federal Reserve.

Tables 2.1–2.4 reveal a more complex pattern of non-monotonicity than under RE in Table 1.1 with several reversals in the effect of policy. In each of the tables one notes that, confining discussion to the region of determinacy, there are three regions which are delineated by either the darker ridge on the right or by the minimum (with respect to ξ_y) region on the left. For example, in Region 1 defined for σ_π in Table 2.2 by approximately [−0.2, 0], as ξ_y rises from −0.2 the volatility of inflation *declines*. In Region 2, defined between the minimal points of σ_y and up to the "ridge" extending from $\xi_y=0.6$ to $\xi_y=30$, output volatility *rises*. In Region 3, beyond the ridge to the right, the volatility *declines* again with ξ_y. The same three Regions are defined for inflation and individual consumption but both the region of minimum volatility and the ridge of maximal volatility vary among the three variables. Observe that in Region 2 an aggressive output stabilization policy using larger values of ξ_y is *self defeating* since it increases the volatility of output rather than decrease it. For individual consumption the curve of minimal σ_c (with respect to ξ_y) extends across Table 2.3.

Table 2.4 provides the example of changed sign of the Equilibrium Determinant we promised earlier. It is seen that the maximal "ridge" of output and inflation volatilities correspond exactly to the values of policy parameters for which the determinant changes sign. It is thus clear that the non-monotonicity phenomenon discussed here is associated with the behavior of the determinant over the feasible policy space.

We dispose of Region 1 which is clearly inefficient; no policy would be selected there since all three volatilities fall in that region. The key differences are between Regions 2 and 3. Before examining policy differences between these last two regions, consider the effect of increasing the value of ξ_π on the three endogenous variables. This effect clearly depends upon ξ_y. In Region 3 the effect of increased ξ_π is to *increase* the volatility of output, inflation and individual consumption. In that region the policy maker selects ξ_π small in the region of determinacy (say $\xi_\pi=1.1$) and employs ξ_y to stabilize the market. In Region 2 the effect of increasing ξ_π is drastically different since

- it reduces the volatility of output,
- it reduces the volatility of inflation, and
- it reduces the volatility of individual consumption *up to a minimum level*, marked in Table 2.3 by italics and dark shading, showing there is a limit to benefits of aggressive anti-inflation policy.

The difference between Regions 2 and 3 is that in Region 2 stabilization of output and inflation is achieved by an aggressive anti-inflation policy according to which the central bank selects ξ_π at the largest politically feasible value. Such policy cannot reduce output volatility beyond a lower bound but can reduce inflation volatility down to zero. Region 3 is characterized by a central bank that aims to lower output volatility to a level which is lower than the lower bound in Region 2, since in Region 3 *output volatility can be reduced to zero* by an aggressive ξ_y policy.

We examined the economy of Table 2 also with a policy that targets the gap $\hat{x}_t=\hat{y}_t-\hat{y}_t^f$, discussed in Section 6.1. We can report that, as one would expect, the quantitative results are slightly different but the qualitative results regarding non-monotonicity and different regions of reaction to policy are the same.

Turning now to a comparison of Regions 2 and 3, observe that an aggressive anti-inflationary policy in Region 2 aims to control volatility caused, in part, by market belief. The policy has two limitations. The first, which we have already noted, is that it has a bounded effect on output volatility, as seen at the bottom of Table 2.1. As $\xi_\pi \Rightarrow \infty$, the value of σ_y is bounded away from zero (0.98 in Table 2.1). To reduce output volatility below this bound, the policy must move to Region 3. In this region aggressive output stabilization policy is effective and large values of ξ_y do suppress output volatility but at the cost of very high volatility of inflation and individual consumption. This volatility is too high and we comment on it later. Confining ourselves, for the moment, to output and inflation, the tables show a limited trade-off between σ_y and σ_π in the conventional sense of being defined on a concave surface. Instead, in this model a trade-off between inflation and output volatilities takes place by a choice *between regions* of the policy space which reflect choice between aggressive inflation stabilization in Region 2 and output stabilization policies in Region 3. These are two different but efficient visions of central bank policy. More generally, a trade-off faced by a central bank in our economy is a complex choice over regions in Tables 2.1–2.3 rather than a smooth choice over a continuous surface.

The second limitation of an anti-inflationary policy in Region 2 is unique to heterogenous economies and does not exist in a single agent economy: the volatility of financial markets and *individual* consumption. Table 2.3 reports the volatility of individual consumption σ_c. This volatility cannot be computed from the macro-model; it must be computed from the microeconomic equilibrium in which individual agents are symmetric. These agents hold diverse beliefs and borrow or lend to act upon these beliefs. Monetary policy has an impact on their choices and fluctuating interest rates interact with private

Table 2.1
Output volatility in the two shocks (ζ, u) model with diverse beliefs.

σ_y		ξ_y													
		−0.2	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**1.2**	**1.4**	**1.6**	**1.8**	**2**	**2.5**	**30**
ξ_π	1.1	2.03	2.84	4.54	9.83	269.44	10.22	5.47	3.83	3.00	2.49	2.15	1.90	1.49	0.15
	1.2	1.54	1.75	2.06	2.52	3.21	4.35	6.50	12.05	59.28	22.27	9.72	6.35	3.54	0.23
	1.3	1.37	1.48	1.61	1.80	2.04	2.36	2.79	3.37	4.22	5.55	7.92	13.31	23.92	0.31
	1.4	1.29	1.35	1.43	1.53	1.66	1.82	2.02	2.26	2.57	2.95	3.46	4.13	7.60	0.39
	1.5	1.23	1.27	1.32	1.39	1.47	1.57	1.69	1.83	2.00	2.19	2.43	2.72	3.77	0.47
	1.75	1.15	1.17	1.19	1.22	1.26	1.31	1.36	1.41	1.48	1.55	1.64	1.73	2.02	0.68
	2	1.11	1.12	1.13	1.15	1.17	1.20	1.22	1.26	1.29	1.33	1.38	1.43	1.57	0.91
	30	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	1.02

Table 2.2
Inflation volatility in the two shocks (ζ, u) model with diverse beliefs.

σ_π		ξ_y													
		−0.2	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**1.2**	**1.4**	**1.6**	**1.8**	**2**	**2.5**	**30**
ξ_π	1.1	5.56	12.23	26.53	71.68	2333	101.73	61.28	47.67	40.93	36.96	34.38	32.58	29.88	25.81
	1.2	2.17	3.76	6.29	10.04	15.79	25.30	43.52	91.03	497.24	205.01	97.30	68.57	45.13	25.63
	1.3	1.48	2.06	3.12	4.60	6.56	9.15	12.63	17.48	24.57	35.80	55.95	102.09	218.91	25.51
	1.4	1.24	1.46	2.01	2.81	3.84	5.13	6.71	8.67	11.14	14.31	18.47	24.12	53.48	25.45
	1.5	1.11	1.20	1.51	2.00	2.64	3.43	4.37	5.48	6.80	8.38	10.29	12.61	21.31	25.44
	1.75	0.92	0.91	1.01	1.20	1.48	1.82	2.22	2.68	3.20	3.79	4.44	5.18	7.46	25.68
	2	0.79	0.77	0.81	0.91	1.06	1.26	1.49	1.75	2.04	2.37	2.72	3.11	4.24	26.29
	30	0.04	0.04	0.05	0.05	0.05	0.05	0.06	0.06	0.07	0.07	0.08	0.08	0.10	1.06

Table 2.3
Individual consumption volatility in the two shocks (ζ, u) model with diverse beliefs.

σ_c		ξ_y													
		−0.2	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**1.2**	**1.4**	**1.6**	**1.8**	**2**	**2.5**	**30**
ξ_π	1.1	2.38	4.43	9.10	24.08	775.25	33.59	20.14	15.61	13.36	12.02	11.15	10.54	9.60	7.83
	1.2	1.44	1.76	2.43	3.56	5.40	8.52	14.55	30.33	165.37	68.09	32.28	22.72	14.90	7.85
	1.3	1.32	1.37	1.53	1.86	2.38	3.15	4.25	5.82	8.16	11.87	18.57	33.90	72.81	7.88
	1.4	***1.30***	1.29	1.32	1.42	1.61	1.91	2.34	2.92	3.69	4.70	6.06	7.92	17.67	7.92
	1.5	1.31	***1.29***	***1.28***	1.30	1.36	1.48	1.67	1.94	2.30	2.76	3.35	4.09	6.93	7.98
	1.75	1.34	1.32	1.29	***1.27***	***1.25***	***1.25***	1.27	1.30	1.36	1.46	1.59	1.76	2.38	8.20
	2	1.37	1.35	1.32	1.30	1.28	1.26	***1.25***	***1.24***	***1.24***	***1.25***	***1.27***	***1.31***	***1.49***	8.52
	30	1.50	1.50	1.50	1.50	1.50	1.49	1.49	1.49	1.49	1.49	1.49	1.49	1.49	1.34

Table 2.4
Equilibrium determinant ($|M| \times 10^3$) in the two shocks (ζ, u) model with diverse beliefs.

$\|M\| \times 10^3$		ξ_y													
		−0.2	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**1.2**	**1.4**	**1.6**	**1.8**	**2**	**2.5**	**30**
ξ_π	**1.1**	0.19	0.17	0.13	0.07	0.00	−0.10	−0.21	−0.35	−0.52	−0.71	−0.94	−1.19	−1.97	−1.E+03
	1.2	0.69	0.68	0.65	0.61	0.55	0.46	0.35	0.21	0.05	−0.15	−0.37	−0.63	−1.43	−1.E+03
	1.3	1.57	1.59	1.60	1.58	1.54	1.47	1.38	1.26	1.11	0.93	0.72	0.47	−0.32	−1.E+03
	1.4	2.94	3.01	3.05	3.08	3.07	3.04	2.98	2.89	2.77	2.62	2.43	2.20	1.45	−1.E+03
	1.5	4.90	5.02	5.12	5.20	5.24	5.26	5.25	5.20	5.12	5.00	4.85	4.66	3.98	−1.E+03
	1.75	12.98	13.30	13.59	13.86	14.09	14.28	14.44	14.56	14.64	14.68	14.68	14.63	14.29	−1.E+03
	2	26.81	27.42	27.99	28.52	29.02	29.48	29.89	30.26	30.59	30.86	31.09	31.27	31.48	−1.E+03
	30	4.E+05	4.E+05	4.E+05	4.E+05	4.E+05	4.E+05	4.E+05	4.E+05	4.E+05	4.E+05	4.E+05	4.E+05	4.E+05	5.E+05

Note: In region below the bold lines the parameters satisfy the Blanchard–Kahn conditions.

Table 3
Ratio of wage volatility to output volatility. Model specification as in Table 2.

σ_w/σ_y		ξ_y					
		0.0	0.2	0.4	0.6	0.8	1.00
ξ_π	1.1	1.74	1.82	1.89	1.91	1.92	1.93
	1.2	1.56	1.65	1.85	1.87	1.89	1.90
	1.4	1.37	1.43	1.77	1.80	1.83	1.85
	2.0	1.15	1.18	1.56	1.61	1.65	1.69
	3.0	1.05	1.06	1.07	1.08	1.09	1.11

expectations to create a difference between volatility of *individual* and *aggregate* consumption the agent is not able to insure against.[9] Note the semi-diagonal configurations in Table 2.3 where σ_c is minimized. One can show that such a minimum occurs when volatility of the real rate is minimized, but an aggressive monetary policy to stabilize (σ_y, σ_π) does not aim to reduce volatility of the real rate. Indeed, volatility of the real rate is a key tool of inflation stabilization. Table 2.3 shows that a policy to stabilize the real rate requires a balance between ξ_π and ξ_y. Consequently, one can see that a central bank that selects $\xi_y = 0$ and stabilizes inflation with a large value of ξ_π will end up increasing the volatility of individual consumption. Welfare considerations imply that monetary policy faces a trade-off among $(\sigma_y, \sigma_\pi, \sigma_c)$: volatility of output, inflation and individual consumption! Since σ_c results from high volatility in the bond market, one must consider σ_c not only as volatility of individual consumption but also as volatility of financial markets in general. It is an undisputed fact that all central banks are concerned with volatility of financial markets. This discussion also questions the view that fluctuations of *aggregate* consumption are the only problem of stabilization policy, as demonstrated by the pointless discussion that followed Lucas (1987, 2003).

Without exhibiting an additional table one can see that the volatility of individual consumption is associated with volatility of borrowing and bond holdings. It explains that political resistance to aggressive use of anti-inflation instrument ξ_π is rooted in the fact that aggressive use of ξ_π entails volatile financial markets and high volatility of bond holdings and individual consumption. Moreover, an examination of Tables 2.1–2.3 reveals that efficient policies are, therefore, moderate in attaining a balance between the desire for low volatility of the real rate and strong inflation stabilization.

High inflation volatility in Region 3, Table 2.2, suggests that no policy would be selected in that region. These inflation rates are excessive by a large factor and result from high wage volatility, assumed fully flexible. Wage flexibility is a problem in all New Keynesian Models that make this assumption. To see one result of the assumption consider Table 3 where we report, for the model specifications of Table 2.1, the ratio of wage volatility to output volatility. The empirical record in industrial countries is similar to the US where $(\sigma_w/\sigma_y = 0.38)$. The table shows model simulations result in a ratio, in Region 3, *which is 5 times larger*. This result remains true for RE and does not depend upon the diverse belief assumption. It is not a surprising result, given the fact that wages are stickier than prices. Some attempts to introduce sticky wages into the New Keynesian Model (e.g. Woodford, 2003; Blanchard and Galí, 2010; Gertler et al., 2008) offer an unsatisfactory view of a labor market with unions or workers setting wages. The problem of inflexible wages and involuntary unemployment in a New Keynesian Model is an important open problem.

Some conclusions from the simulations

The simulation offers a sample of the implications of a theory of diverse beliefs to the conduct of monetary policy. Although the results in Tables 2.1–2.4 vary with model parameters, the qualitative conclusions continue to hold. The main conclusion is that in an economy with diverse beliefs the effect of policy instruments is non-monotonic, with complex thresholds which may be difficult for a central bank to compute with precision. It also suggests that

- output stabilization is a difficult task which competes with the objective of stable financial markets and low volatility of individual consumption. It is thus not surprising that differences exist among central banks with respect to the goal of output stabilization,
- trade-off between policy objectives may not take place on a smooth concave surface but rather as a choice between regions of the policy space, each representing an efficient central bank policy, and
- New Keynesian Models with flexible wages generate inflation volatility which is much too high.

[9] The reader may find elsewhere (see Kurz, 2010 and Kurz and Motolese (2011)) detailed explanation of why it is not incentive compatible to have markets for claims which are contingent on future market belief. Here we note briefly that market belief must be computed using data on surveys of individual forecasts and the existence of such markets will create a public motive to distort the reported forecasts. The portion of the population which is short will have an incentive to report so as to lead to computed low level of market belief and the portion which is long will have the incentive to report the opposite. No court can rely upon such information to resolve legal disputes about financial obligations. Due to market incompleteness agents cannot reduce consumption volatility by trading in markets for contingent claims.

7. Some comments on forward looking rules

We now examine forward looking rules like $\hat{r}_t = \xi_\pi E_t[\hat{\pi}_{t+1}] + \xi_y E_t[\hat{y}_{t+1}]$ but for simplicity study a one shock model, setting $u_t = 0$. With diverse beliefs a natural question arises: which expectations are to be used? Two answers come to mind. First, a central bank can use the mean market belief and employ the rule

$$\hat{r}_t = \xi_\pi \overline{E}_t[\hat{\pi}_{t+1}] + \xi_y \overline{E}_t[\hat{y}_{t+1}] \tag{37a}$$

Second, the central bank may use *its own belief model* which we can denote by *cb* and write as

$$\hat{r}_t = \xi_\pi E_t^{cb}[\hat{\pi}_{t+1}] + \xi_y E_t^{cb}[\hat{y}_{t+1}] \tag{37b}$$

These two have different implications. The key question is how the policy rule alters the equilibrium map. In much of the monetary policy literature this issue is resolved by rewriting the *macro-model* and then deducing a reduced form equilibrium solution of the difference equations by forward iterations (see Blanchard and Kahn, 1980). With diverse beliefs this procedure cannot be employed without first solving two problems. First, the average market expectation operator $\overline{E}_t$ is not a conditional expectation of a proper probability and forward iterations cannot be carried out. Second one must solve the *micro-equilibrium* from which to deduce missing parameters of the macro-economy. Neither problem can be solved if equilibrium state variables are not explicitly clarified and these must now be checked for the two cases noted above.

7.1. Equilibrium under $\hat{r}_t = \xi_\pi \overline{E}_t[\hat{\pi}_{t+1}] + \xi_y \overline{E}_t[\hat{y}_{t+1}]$

Individual optimum conditions (3a)–(3c) and (14) r_t take as given. Hence, a perspective of dynamic optimization implies that the agent's state variables are $(Z_t, \hat{\zeta}_t, \hat{b}^j_{t-1}, g^j_t)$ as before, except for the fact that an agent must forecast interest rates. Market clearing conditions ensure that $\hat{b}^j_{t-1}$ is aggregated to zero and is not a macro-state variable. Also, since in the log linearized economy the aggregates $(\hat{\pi}_t, \hat{y}_t)$ are linear in state variables, forecasting $\hat{r}_{t+1}$ requires averaging forecasts of $(\hat{\pi}_{t+1}, \hat{y}_{t+1})$ by all others. But averaging $(Z_t, \hat{\zeta}_t, \hat{b}^j_{t-1}, g^j_t)$ yields the two state variables $(Z_t, \hat{\zeta}_t)$ as in the case of a policy rule $\hat{r}_t = \xi_\pi \hat{\pi}_t + \xi_y \hat{y}_t$. An alternative and a simpler argument is to assume that the micro-state variables are $(Z_t, \hat{\zeta}_t, \hat{b}^j_{t-1}, g^j_t)$ and the macro state variables $(Z_t, \hat{\zeta}_t)$ and simply check directly for consistency. In either case we have

Proposition 8. If the policy rule is $\hat{r}_t = \xi_\pi \overline{E}_t[\hat{\pi}_{t+1}] + \xi_y \overline{E}_t[\hat{y}_{t+1}]$ the results of Theorem 2 remain valid: equilibrium is regular with finite memory and the policy rule can be transformed into

$$\hat{r}_t = \xi_\pi E_t^m[\hat{\pi}_{t+1}] + \xi_y E_t^m[\hat{y}_{t+1}] + \Gamma^r Z_t,\ \Gamma^r = \xi_y \Gamma^\pi + \xi_y \Gamma^y$$

The conditions for determinacy are, however, different.

For two determinacy conditions that apply to this case see Galí (2008), p. 79. Using the theorem above one can now rewrite the system (16a)–(16c) in the form

$$\text{IS Curve} \quad \hat{y}_t = E_t^m(\hat{y}_{t+1}) + B_y Z_t - \left(\frac{1}{\sigma}\right)\left[r_t - E_t^m(\hat{\pi}_{t+1})\right] \tag{38a}$$

$$\text{Phillips curve} \quad \hat{\pi}_t = \kappa(\eta+\sigma)\hat{y}_t + \beta E_t^m \hat{\pi}_{t+1} + \beta B_\pi Z_t - \kappa(1+\eta)\hat{\zeta}_t \tag{38b}$$

$$\text{Monetary rule} \quad \hat{r}_t = \xi_\pi E_t^m[\hat{\pi}_{t+1}] + \xi_y E_t^m[\hat{y}_{t+1}] + \Gamma^r Z_t, \quad \Gamma^r = \xi_\pi \Gamma^\pi + \xi_y \Gamma^y \tag{38c}$$

with transitions of $(\hat{\zeta}_t, Z_t)$. The results are easily extended to allow any other shocks.

A forward looking monetary rule has been correctly justified on sound grounds that we do not review here. But a price is paid for using a rule based on forecasts. It is an *added volatility induced by the rule itself*. In (38c) one can see it in the added term $\Gamma^r Z_t$ which amplifies volatility. It reflects uncertainty of future belief employed by the policy. It is useful to stress a general principle: *with diverse beliefs, a bank's decisions based on forecasts trigger diverse views about future beliefs employed in such forecasts and this diversity amplifies volatility*. A forward looking rule thus entails adding to that same volatility which the rule aims to stabilize! Note that a central bank can reduce its own effect on volatility by using the empirical probability m as its belief. Such a credible decision by the bank will eliminate the term $\Gamma^r Z_t$ in (36c).

7.2. Equilibrium under $\hat{r}_t = \xi_\pi E_t^{cb}[\hat{\pi}_{t+1}] + \xi_y E_t^{cb}[\hat{y}_{t+1}]$

When a central bank uses its own forecasting model the situation changes. A bank is just another agent with its own belief among rational agents and private agents do not consider the bank's belief as superior. If the bank has a credible policy in place then it does not have any information which the public does not possess since no one has private information about the macro-economy. The belief of the central bank is public in the same way average private belief is public information. The ability of a bank to commit to a policy is naturally an important question and so far this paper's analysis was conducted by assuming a central bank can commit to a policy rule. But what is the confidence of the private sector in the forecast ability of the central bank? Empirical evidence suggests a central bank does not forecast GDP growth or inflation with great precision. Using the Green Book forecasts by the Federal Reserve one can correlate forecasts with realization and compute the R^2 between the two. Such computations reveal the Fed's forecast accuracy of GDP growth for the “present” quarter is low with

R^2 of about 0.5. For longer horizons they are very unreliable with R^2 of about 0.25 for one quarter and R^2 that fall to around 0.15 for horizons above 6 months. The standard errors of the forecasts are usually very large. Inflation is highly persistent hence forecasting it is much easier. Even here the typical R^2 between forecasts and realization by the Fed is about 0.5 for horizons longer than 6 months.

Two additional facts about the Fed forecasting may be mentioned. First, the Green Book forecasts are made by the *Federal Reserve staff* and released five years later. If the aim is to inform the public about the Fed's views why are the forecasts released only five years later? Second, members of the open market committee make their own individual forecasts and these are released with the committee's minutes. Examination of these reveals wide differences in forecasts of GDP growth and inflation among members even for relatively short horizons. With wide differences of forecasts *within* the Fed, any official "Central Bank Forecast" can only be some sort of a compromise the nature of which will only trigger more market speculations. It is thus not surprising markets treat the Fed forecasts as just one more forecast made by just one more agent. An important agent, to be sure, but not different in structure from the belief of anyone else.

To model the central bank as an agent requires us to specify the belief index g_t^{cb} of the central bank. Using the same logic as the private sector, one establishes the transition of g_t^{cb} to be as in (25c) and express the belief model of the bank by $(\hat{\zeta}_{t+1}^{cb}, Z_{t+1}^{cb}, g_{t+1}^{cb})$ with transitions

$$\hat{\zeta}_{t+1}^{cb} = \lambda_\zeta \hat{\zeta}_t + \lambda_\zeta^{gcb} g_t^{cb} + \rho_{t+1}^{cb\zeta}, \tag{39a}$$

$$Z_{t+1}^{cb} = \lambda_Z Z_t + \lambda_Z^\zeta [\hat{\zeta}_{t+1} - \lambda_\zeta \hat{\zeta}_t] + \lambda_Z^{gcb} g_t^{cb} + \rho_{t+1}^{cbZ}, \tag{39b}$$

$$g_{t+1}^{cb} = \lambda_Z g_t^{cb} + \lambda_Z^\zeta [\hat{\zeta}_{t+1} - \lambda_\zeta \hat{\zeta}_t] + \rho_{t+1}^{cbg}, \tag{39c}$$

$$\begin{pmatrix} \rho_{t+1}^{cb\zeta} \\ \rho_{t+1}^{cbZ} \\ \rho_{t+1}^{cbg} \end{pmatrix} \sim N \left(\begin{matrix} 0 \\ 0, \\ 0 \end{matrix} \begin{bmatrix} \hat{\sigma}_\zeta^2, & 0, & 0 \\ 0, & \hat{\sigma}_Z^2, & \hat{\sigma}_{Zg}, \\ 0, & \hat{\sigma}_{Zg}, & \hat{\sigma}_g^2 \end{bmatrix} \right)$$

and a normalization $\lambda_\zeta^{gcb} = 1$. Eq. (39c) is the empirical distribution of g_t^{cb}. Agents take g_t^{cb} as a new and relevant state variable hence agent j's vector of state variables becomes $(Z_t, \hat{\zeta}_t, \hat{b}_{t-1}^j, g_t^j, g_t^{cb})$. One then recognizes the profound effect of the new state variable which triggers private sector speculations about its future evolution. That is, is it true that (37c) is the true transition of g_t^{cb}? With individual beliefs formed about future beliefs of the central bank, the belief structure of the private sector is further complicated. In this case private belief change from (25a)–(25c) to a general form of perception by private agent i

$$\hat{\zeta}_{t+1}^i = \lambda_\zeta \hat{\zeta}_t + \lambda_\zeta^g g_t^i + \rho_{t+1}^{i\zeta} \tag{40a}$$

$$Z_{t+1}^i = \lambda_Z Z_t + \lambda_Z^\zeta [\hat{\zeta}_{t+1} - \lambda_\zeta \hat{\zeta}_t] + \lambda_Z^g g_t^i + \rho_{t+1}^{iZ} \tag{40b}$$

$$g_{t+1}^{cbi} = \lambda_Z g_t^{cb} + \lambda_Z^\zeta [\hat{\zeta}_{t+1} - \lambda_\zeta \hat{\zeta}_t] + \lambda_{cb}^g g_t^i + \rho_{t+1}^{cbg} \tag{40c}$$

$$g_{t+1}^i = \lambda_Z g_t^i + \lambda_Z^\zeta [\hat{\zeta}_{t+1} - \lambda_\zeta \hat{\zeta}_t] + \rho_{t+1}^{ig} \tag{40d}$$

with a covariance matrix. In (40a)–(40d) a belief state g_t^i impacts three perceived transitions of macro-state variables $(\hat{\zeta}_t, Z_t, g_t^{cb})$ taken exogenously by i. Both private agents and the central bank formulate belief about future business conditions expressed by values of $\hat{\zeta}_{t+1}$. This triggers an expanded individual state space and belief about future market belief Z_{t+1} and future central bank belief g_{t+1}^{cb} with parameters $(\lambda_Z^g, \lambda_{cb}^g)$. Note the general principle implied: an ambiguity about the future leads to an expansion of the issues subject to diverse belief and further amplification of market volatility. By basing policy on its own belief, a central bank opens the door for the market to endogenously add a component of uncertainty which was not there before.

Aggregation and market clearing conditions show that macro-state variables $(Z_t, \hat{\zeta}_t, g_t^{cb})$ are. In this case an equilibrium with the central bank as an agent becomes *regular with finite memory* hence using (37a)–(37c) one can compute the new policy rule as follows. First compute the differences

$$E_t^{cb}[\hat{\pi}_{t+1}] - E_t^m[\hat{\pi}_{t+1}] = [(A_\pi^Z + A_\pi^g)\lambda_Z^\zeta + A_\pi^\zeta + A_\pi^{gcb} \lambda_Z^{gcb}] g_t^{cb}$$

$$E_t^{cb}[\hat{y}_{t+1}] - E_t^m[\hat{y}_{t+1}] = [(A_y^Z + A_y^g)\lambda_Z^\zeta + A_y^\zeta + A_y^{gcb} \lambda_Z^{gcb}] g_t^{cb}.$$

Next, write the new policy rule for the linearized economy expressed in terms of the probability m

$$\hat{r}_t = \xi_\pi E_t^{cb}[\hat{\pi}_{t+1}] + \xi_y E_t^{cb}[\hat{y}_{t+1}] = \xi_\pi E_t^m[\hat{\pi}_{t+1}] + \xi_y E_t^m[\hat{y}_{t+1}] + [\xi_\pi \Gamma^{\pi cb} + \xi_y \Gamma^{ycb}] g_t^{cb} \tag{41}$$

where

$$\Gamma^{\pi cb} = [(A^Z_\pi + A^g_\pi)\lambda^\zeta_Z + A^\zeta_\pi + A^{gcb}_\pi \lambda^{gcb}_Z], \quad \Gamma^{ycb} = [(A^Z_y + A^g_y)\lambda^\zeta_Z + A^\zeta_y + A^{gcb}_y \lambda^{gcb}_Z],$$

But recall that the equilibrium parameters $(A^Z_\pi, A^g_\pi, A^\zeta_\pi, A^{gcb}_\pi, A^Z_y, A^g_y, A^\zeta_y, A^{gcb}_y)$ depend upon the policy!

The conclusion we draw from (36a) to (36c) and (39) is that, in the case at hand, when the central bank's belief is Markov and the empirical probability m is Markov, the resulting equilibrium has the same analytical structure as (32a)–(32c). Different policies will surely exhibit different dynamic properties but the causal structure remains the same: market belief *does not have an effect on determinacy*, and Theorem 2 and Propositions 3 and 4 continue to hold. Since this macro system has different parameters the conditions for determinacy are different but employ the same formula (see Galí, (2008), page79).

Appendix A. Proofs of Theorem 1 and of Propositions 4 and 5

Proof of Theorem 1

Assumption 3. The alternate prior, based on the public signal, incorporates a direct learning process where

$$S^i_t \sim N\left(\psi^i_t, \frac{1}{\gamma}\right)$$

One interpret ψ^i_t as a prior subjective mean, positive or negative, *given the qualitative public signal.*

Proof. Using Assumption 3 combine the two sources to have that

$$s_t(\hat{\zeta}_t, \psi^i_t) = \mu s_{t-1}(\hat{\zeta}_t) + (1-\mu) S^i_t$$

with a mean of $E^i_t(s_t \big| \hat{\zeta}_t, \psi^i_t) = \mu E^i_t(s_{t-1} \big| \hat{\zeta}_t) + (1-\mu)\psi^i_t \quad 0 < \mu < 1$

and conditional variance $Var(s_t | \hat{\zeta}_t, \psi^i_t) = \dfrac{\mu^2}{(\alpha+\nu)} + \dfrac{(1-\mu)^2}{\gamma}$

Let $\xi_1 = (1/\mu^2)$ and $\xi_2 = (1/(1-\mu)^2)$ and we write the precision of the distribution of this new posterior as

$$\text{Precision}(s_t | \hat{\zeta}_t, \psi^i_t) = \Gamma(s_t) = 1 \Big/ \left[\frac{1}{\xi_1(\alpha+\nu)} + \frac{1}{\xi_2 \gamma}\right] = \frac{\xi_1(\alpha+\nu)\xi_2\gamma}{\xi_1(\alpha+\nu)+\xi_2\gamma}$$

At date $t+1$ the agent observes $\hat{\zeta}_{t+1}$. By (18a) in the text in the form $\hat{\zeta}_{t+1}$-$\lambda_\zeta \hat{\zeta}_t = s_t + \varepsilon^\zeta_{t+1}$, $\varepsilon^\zeta_{t+1} \sim N(0, \frac{1}{\nu})$ it follows that updating $E^i_t(s_t | \hat{\zeta}_t, \psi^i_t)$ the agent has

$$E^i_{t+1}(s_t | \hat{\zeta}_{t+1}, \psi^i_t) = \frac{\Gamma(s_t) E^i_t(s_t | \hat{\zeta}_t, \psi^i_t) + \nu[\hat{\zeta}_{t+1} - \lambda_\zeta \hat{\zeta}_t]}{\Gamma(s_t)+\nu},$$

$$s_t(\hat{\zeta}_{t+1}, \psi^i_t) \sim N\left[E^i_t(s_t | \hat{\zeta}_{t+1}, \psi^i_t), \frac{1}{\Gamma(s_t)+\nu}\right].$$

After assessing the mean ψ^i_{t+1} he formulates the new posterior which is

$$s_{t+1}(\hat{\zeta}_{t+1}, \psi^i_{t+1}) = \mu s_t(\hat{\zeta}_{t+1}, \psi^i_t) + (1-\mu) S^i_{t+1}$$

with mean

$$E^i_{t+1}(s_{t+1} | \hat{\zeta}_{t+1}, \psi^i_{t+1}) = \mu E^i_{t+1}(s_t | \hat{\zeta}_{t+1}, \psi^i_t) + (1-\mu)\psi^i_{t+1} \quad 0 < \mu < 1 \tag{A.1}$$

conditional variance

$$Var(s_{t+1} | \hat{\zeta}_{t+1}, \psi^i_{t+1}) = \frac{1}{\xi_1(\Gamma(s_t)+\nu)} + \frac{1}{\xi_2\gamma} \tag{A.2}$$

and precision

$$\Gamma(s_{t+1}) = 1 \Big/ \left[\frac{1}{\xi_1(\Gamma(s_t)+\nu)} + \frac{1}{\xi_2\gamma}\right] = \frac{\xi_1(\Gamma(s_t)+\nu)\xi_2\gamma}{\xi_1(\Gamma(s_t)+\nu)+\xi_2\gamma} \tag{A.3}$$

We can now deduce the full symmetry of the process. For large t we then have

$$s_t(\hat{\zeta}_{t+1}, \psi^i_t) \sim N\left[E^i_t(s_t | \hat{\zeta}_{t+1}, \psi^i_t), \frac{1}{\Gamma(s_t)+\nu}\right]$$

$$E^i_t(s_t | \hat{\zeta}_{t+1}, \psi^i_t) = \frac{\Gamma(s_t) E^i_t(s_t | \hat{\zeta}_t, \psi^i_t) + \nu[\hat{\zeta}_{t+1} - \lambda_\zeta \hat{\zeta}_t]}{\Gamma(s_t)+\nu}$$

After observing ψ^i_{t+1} the new posterior is

$$s_{t+1}(\hat{\zeta}_{t+1},\psi^i_{t+1}) = \mu s_t(\hat{\zeta}_{t+1},\psi^i_t)+(1-\mu)S^i_{t+1}$$

The mean, conditional variance and precision are then as in (A1)–(A.3) and hence we have an equation for the precision

$$\Gamma_{t+1} = \frac{\xi_1(\Gamma_t+\nu)\xi_2\gamma}{\xi_1(\Gamma_t+\nu)+\xi_2\gamma}$$

It is well defined for $1<\xi_1<\infty$ (i.e. $0<\mu<1$) and in that case it has the unique positive solution

$$\xi_1 = \frac{1}{\mu^2} \quad \text{and} \quad \xi_2 = \frac{1}{(1-\mu)^2}, \quad \Gamma^* = \frac{(\nu+\xi_2\gamma(1-(1/\xi_1)))+\sqrt{(\nu+\xi_2\gamma(1-(1/\xi_1)))^2+4\nu}}{2} > 0$$

The negative root has no economic meaning. When $\xi_1 = 1$, $\xi_2 = \infty$ there is no solution, and Γ_t diverges for large t, which is the classical case. Now insert $\Gamma = \Gamma^*$ into the equations above to deduce that

$$E^i_{t+1}(s_{t+1}|\hat{\zeta}_{t+1},\psi^i_{t+1}) = \mu E^i_{t+1}(s_t|\hat{\zeta}_{t+1},\psi_t)+(1-\mu)\psi^i_{t+1}$$

hence

$$E^i_{t+1}(s_{t+1}|\hat{\zeta}_{t+1},\psi^i_{t+1}) = \mu\frac{\Gamma^* E^i_t(s_t|\hat{\zeta}_t,\psi^i_t)+\nu[\hat{\zeta}_{t+1}-\lambda_\zeta\hat{\zeta}_t]}{\Gamma^*+\nu}+(1-\mu)\psi^i_{t+1} \tag{A.4}$$

Now define

$$g^i_{t+1} = E^i_{t+1}(s_{t+1}|\hat{\zeta}_{t+1},\psi^i_{t+1}),\ \lambda_z = \frac{\mu\Gamma^*}{\Gamma^*+\nu}>0,\ \lambda^\zeta_z = \frac{\mu\nu}{\Gamma^*+\nu}>0,\ \rho^{ig}_{t+1} = (1-\mu)\psi^i_{t+1}$$

Hence, the law of motion of g^i_{t+1} is

$$g^i_{t+1} = \lambda_z g^i_t+\lambda^\zeta_z[\hat{\zeta}_{t+1}-\lambda_\zeta\hat{\zeta}_t]+\rho^{ig}_{t+1} \qquad \square \tag{A.5}$$

Some comments: Aggregation implies an empirical distribution of the form

$$Z_{t+1} = \lambda_z Z_t+\lambda^\zeta_z[\hat{\zeta}_{t+1}-\lambda_\zeta\hat{\zeta}_t]+\rho^z_{t+1} \tag{A.6}$$

and a belief of i in the mean belief of others is defined by

$$Z^i_{t+1} = \lambda_z Z_t+\lambda^\zeta_z[\hat{\zeta}_{t+1}-\lambda_\zeta\hat{\zeta}_t]+\lambda^g_z g^i_t+\rho^{iz}_{t+1}$$

For simulations one uses expressions (A.5) and (A.6). General equilibrium computations are based on expectations of (A.5) and (A.6) which are

$$E^i_t\left(g^i_{t+1}\right) = (\lambda_z+\lambda^\zeta_z)g^i_t$$
$$E^i_t\left(Z^i_{t+1}\right) = \lambda_z Z_t+(\lambda^\zeta_z+\lambda^g_z)g^i_t$$

These are then used in the general equilibrium computations of (A_y, A_π) In addition, we have that

$$\int_0^1 \left[E^i_t\left(g^i_{t+1}\right)-E^i_t\left(Z^i_{t+1}\right)\right]dt = \lambda^g_z Z_t \tag{A.7}$$

Proof of Proposition 4. It is explained in Appendix B that equilibrium values (A_y, A_π, A_b) of the micro-model are deduced from the log-linearized Euler Eqs. (5a) and (14) which we write in the form

$$\hat{c}^j_t+\left(\frac{1}{\sigma}\right)[\xi_\pi\hat{\pi}_t+\xi_y\hat{y}_t] = E^j_t\left(\hat{c}^j_{t+1}\right)+\left(\frac{1}{\sigma}\right)E^j_t(\hat{\pi}_{t+1})+\tau_b\hat{b}^j_t$$
$$\frac{1-\omega}{\omega}\hat{q}^*_{jt} = -\kappa(1+\eta)\hat{\zeta}_t+\kappa(\eta+\sigma)\hat{y}_t+\beta(1-\omega)E^j_t\left[\hat{q}^*_{j(t+1)}+\hat{\pi}_{t+1}\right]$$

By (31a)–(31f) one writes these equations in the following linear form in j's expected values

$$A_y\bullet(Z_t,\hat{\zeta}_t,\hat{b}^j_{t-1},g^j_t)+\frac{\xi_\pi}{\sigma}\left[A_\pi\bullet(Z_t,\hat{\zeta}_t,0,Z_t)\right]+\frac{\xi_y}{\sigma}\left[A_y\bullet(Z_t,\hat{\zeta}_t,0,Z_t)\right]$$
$$= A_y\bullet\left(E^j_t[Z_{t+1}],E^j_t\left[\hat{\zeta}_{t+1}\right],\hat{b}^j_t,E^j_t\left[g^j_{t+1}\right]\right)+\left(\frac{1}{\sigma}\right)A_\pi\bullet(E^j_t[Z_{t+1}],E^j_t\left[\hat{\zeta}_{t+1}\right],0,E^j_t[Z_{t+1}])+\tau_b A_b\bullet(Z_t,\hat{\zeta}_t,\hat{b}^j_{t-1},g^j_t) \tag{A.8a}$$

$$A_\pi\bullet(Z_t,\hat{\zeta}_t,\hat{b}^j_{t-1},g^j_t)+\kappa(\eta+\sigma)A_y\bullet(Z_t,\hat{\zeta}_t,0,Z_t)]$$
$$= -\kappa(1+\eta)\hat{\zeta}_t+\beta\omega A_\pi\bullet(E^j_t[Z_{t+1}],E^j_t[\hat{\zeta}_{t+1}],\hat{b}^j_t,E^j_t[g^j_{t+1}])+\beta(1-\omega)A_\pi\bullet(E^j_t[Z_{t+1}],E^j_t[\hat{\zeta}_{t+1}],0,E^j_t[Z_{t+1}]) \tag{A.8b}$$

and expectations defined by

$$E_t^j[\hat{\zeta}_{t+1}^j] = \lambda_\zeta \hat{\zeta}_t + \lambda_\zeta^g g_t^i$$
$$E_t^j[Z_{t+1}] = \lambda_Z Z_t + \lambda_Z^\zeta g_t^j + \lambda_Z^g g_t^j$$
$$E_t^j[g_{t+1}^j] = \lambda_z g_t^j + \lambda_Z^\zeta g_t^j$$

The 12 equilibrium function values (A_y, A_π, A_b) are determined by matching coefficients of the four state variables across equations. Inserting the expectation values into (A.8a) and (A.8b) one obtains 8 equations in the 12 unknown equilibrium values. It is shown in Appendix B that the final four restrictions follow from the optimal borrowing function which is deduced from the budget constraint of agent j. Under the insurance assumption, this equation is defined by

$$\hat{b}_t^j = \frac{1}{\beta}\hat{b}_{t-1}^j + \left[1 + \frac{\theta-1}{\theta}\frac{\sigma}{\eta}\right](\hat{y}_t - \hat{c}_t^j) \tag{A.8c}$$

Return now to (31a)–(31d) and by matching coefficients one deduces that the final four restrictions are

$$A_b^z = A_y^g\left[1 + \frac{\theta-1}{\theta}\frac{\sigma}{\eta}\right], \quad A_b^\zeta = 0, \quad A_b^b = \frac{1}{\beta} - A_y^b\left[1 + \frac{\theta-1}{\theta}\frac{\sigma}{\eta}\right], \quad A_b^g = -A_y^g\left[1 + \frac{\theta-1}{\theta}\frac{\sigma}{\eta}\right] \tag{A.8d}$$

This procedure cannot be carried out for (31a)–(31d). Using (31d) and (31e) one can write it in a linear form, and even by using the borrowing restriction (A.8c) one can deduce only eight equations in 12 unknowns. □

Proof. of Proposition 5

Matching parameters of the state variable $\hat{b}_{t+1}^i$ in (A.8a) and (A.8b) leads to two non-linear equations

$$A_y^b(1 - A_b^b) = \tau_b A_b^b \tag{A.9a}$$

$$A_\pi^b = \beta\omega A_\pi^b A_b^b \tag{A.9b}$$

Now, $A_y^b = 0 \Rightarrow A_b^b = 0$ due to (A.9a) and $A_b^b = (1/\beta) > 0$ due to (A.8d). This is a contradiction, hence $A_y^b \neq 0$ (A.8d) and (A.9a) imply that

$$A_b^b = \frac{A_y^b}{A_y^b + \tau_b} = \frac{1}{\beta} - A_y^b\left[1 + \frac{\theta-1}{\theta}\frac{\sigma}{\eta}\right] \text{ hence } A_y^b = \left[\frac{1}{\beta} - \frac{A_y^b}{A_y^b + \tau_b}\right]\frac{1}{[1 + ((\theta-1)/\theta)(\sigma/\eta)]} \tag{A.9c}$$

If $A_\pi^b \neq 0$ it follows that $A_b^b = (1/\beta\omega)$ which contradicts (A.9c) hence $A_\pi^b = 0$. This implies that

$$A_\pi^b A_b^z = 0, \; A_\pi^b A_b^\zeta = 0, \; A_\pi^b A_b^g = 0$$

Now use (A.8d) and (A.9c) to deduce

$$A_y^b A_b^z = \left[\frac{1}{\beta} - \frac{A_y^b}{A_y^b + \tau_b}\right] A_y^g$$
$$A_y^b A_b^\zeta = 0$$
$$A_y^b A_b^g = -\left[\frac{1}{\beta} - \frac{A_y^b}{A_y^b + \tau_b}\right] A_y^g$$

Next, let $\Xi = 1 + ((\theta-1)/\theta)(\sigma/\eta)$ which is positive since $\theta > 1$ Then, (A.9c) implies the equation.

$$(A_y^b)^2 - \frac{1}{\Xi}\left(\frac{1-\beta}{\beta} - \tau_b\right)A_y^b - \frac{\tau_b}{\beta\Xi} = 0$$

for which there are two *exact* solutions

$$A_y^b = \frac{(\frac{1}{\Xi})(\frac{1-\beta}{\beta} - \tau_b) \pm \sqrt{\frac{1}{\Xi^2}(\frac{1-\beta}{\beta} - \tau_b)^2 + 4\frac{\tau_b}{\beta\Xi}}}{2} \tag{A.10}$$

one positive and one negative. Indeed, these are approximately

$$A_y^b \cong \frac{1}{\Xi}\left(\frac{1-\beta}{\beta}\right) > 0, \quad A_y^b \cong -\frac{\tau_b}{|\sqrt{\beta\Xi}|} < 0$$

To deduce equilibrium insert a solution of (A.10) into the six products. Eqs. (A.9a) and (A.9b) then imply six linear equations in the six parameters (A_y, A_π). But two solutions of A_y^b imply two solutions for (A_y, A_π, A_b). Since A_y^b measures the effect of bond holdings on consumption, $A_y^b < 0$ imply increased consumption and borrowing when in debt and *this causes individual debt to diverge for any* τ_b. This second solution is thus not an equilibrium! For $\tau_b > 0$ the only equilibrium is the one implied by $A_y^b > 0$ in (A.10). □

Appendix B. Identification of parameters with $u=0$

Decision functions in the log-linearized economy take the following form:

$$\hat{c}_t^j = A_y^z Z_t + A_y^\zeta \hat{\zeta}_t + A_y^b \hat{b}_{t-1}^j + A_y^g g_t^j = A_y \bullet (Z_t, \hat{\zeta}_t, \hat{b}_{t-1}^j, g_t^j) \tag{B.1a}$$

$$\hat{b}_t^j = A_b^z Z_t + A_b^\zeta \hat{\zeta}_t + A_b^b \hat{b}_{t-1}^j + A_b^g g_t^j = A_b \bullet (Z_t, \hat{\zeta}_t, \hat{b}_{t-1}^j, g_t^j) \tag{B.1b}$$

$$\hat{q}_{jt}^* = \frac{\omega}{1-\omega}\left[A_\pi^z Z_t + A_\pi^\zeta \hat{\zeta}_t + A_\pi^b \hat{b}_{t-1}^j + A_\pi^g g_t^j\right] = \frac{\omega}{1-\omega} A_\pi \bullet (Z_t, \hat{\zeta}_t, \hat{b}_{t-1}^j, g_t^j) \tag{B.1c}$$

$$E_t^j[\hat{\zeta}_{t+1}^j] = \lambda_\zeta \hat{\zeta}_t + \lambda_\zeta^g g_t^j \tag{B.1d}$$

$$E_t^j[Z_{t+1}] = \lambda_Z Z_t + \lambda_Z^\zeta g_t^i + \lambda_Z^g g_t^j \tag{B.1e}$$

$$E_t^j[g_{t+1}^j] = \lambda_Z g_t + \lambda_Z^\zeta g_t^i \tag{B.1f}$$

One starts by using the consumption and optimal price decision functions but deduce the borrowing function $\hat{b}_t^j$ from the budget constraint. Hence, write down the two linearized optimal conditions (5a) and (14) to have

$$\hat{c}_t^j + \left(\frac{1}{\sigma}\right)[\xi_\pi \hat{\pi}_t + \xi_y \hat{y}_t] = E_t^j(\hat{c}_{t+1}^j) + \left(\frac{1}{\sigma}\right) E_t^j(\hat{\pi}_{t+1}) + \tau_B \hat{b}_t^j$$
$$\frac{1-\omega}{\omega}\hat{q}_{jt}^* = -\kappa(1+\eta)\hat{\zeta}_t + \kappa(\eta+\sigma)\hat{y}_t + \beta(1-\omega)E_t^j\left[\hat{q}_{j(t+1)}^* + \hat{\pi}_{t+1}\right].$$

These can be written in the linear form implied by (B.1a)–(B.1f) as follows:

$$\begin{aligned}
&\left[A_y \bullet (Z_t, \hat{\zeta}_t, \hat{b}_{t-1}^j, g_t^j)\right] + \frac{\xi_y}{\sigma}\left[A_y \bullet (Z_t, \hat{\zeta}_t, 0, Z_t)\right] + \frac{\xi_\pi}{\sigma}\left[A_\pi \bullet (Z_t, \hat{\zeta}_t, 0, Z_t)\right] \\
&= A_y \bullet \left(\left[\lambda_Z Z_t + \lambda_Z^\zeta g_t^j\right], \left[\lambda_\zeta \hat{\zeta}_t + \lambda_\zeta^g g_t^j\right], A_b \bullet (Z_t, \hat{\zeta}, \hat{b}_{t-1}^j, g_t^j), \left[\lambda_Z g_t^j + \lambda_Z^\zeta g_t^j\right]\right) \\
&\quad + \left(\frac{1}{\sigma}\right) A_\pi \bullet \left(\left[\lambda_Z Z_t + \lambda_Z^\zeta g_t^j + \lambda_Z^g g_t^j\right], \left[\lambda_\zeta \hat{\zeta}_t + \lambda_\zeta^g g_t^j\right], 0, \left[\lambda_Z Z_t + \lambda_Z^\zeta g_t^j + \lambda_Z^g g_t^j\right]\right) + \tau_B A_{\mathbf{B}} \bullet (Z_t, \hat{\zeta}_t, \hat{b}_{t-1}^j, g_t^j) \\
&\quad A_\pi \bullet (Z_t, \hat{\zeta}_t, \hat{b}_{t-1}^j, g_t^j) - \kappa(\eta+\sigma) A_y \bullet (Z_t, \hat{\zeta}_t, 0, Z_t) + \kappa(1+\eta)\hat{\zeta}_t \\
&= \beta\omega A_\pi \bullet ([\lambda_Z Z_t + \lambda_Z^\zeta g_t^j + \lambda_Z^g g_t^j], [\lambda_\zeta \hat{\zeta}_t + \lambda_\zeta^g g_t^j], A_b \bullet (Z_t, \hat{\zeta}_t, \hat{b}_{t-1}^j, g_t^j), [\lambda_Z g_t + \lambda_Z^\zeta g_t^j]) \\
&\quad + \beta(1-\omega) A_\pi \bullet ([\lambda_Z Z_t + \lambda_Z^\zeta g_t^i], [\lambda_\zeta \hat{\zeta}_t + \lambda_\zeta^g g_t^j], 0, [\lambda_Z Z_t + \lambda_Z^\zeta g_t^j + \lambda_Z^g g_t^j])
\end{aligned}$$

Given parameters A_b one matches coefficients to have 8 equations deduced from each of the equations above, in the 8 unknown parameters (A_y, A_π). But to carry this out we need the borrowing function with the penalty on excessive borrowing.

To compute A_b from the budget constraint, the budget to be used is the one deduced from the insurance assumption. That is, the *effective budget which takes into account the transfers*

$$C_t^j + \frac{M_t^j}{P_t} + \frac{B_t^j}{P_t} = \left(\frac{W_t}{P_t}\right) L_t^j + \left[\frac{B_{t-1}^j(1+r_{t-1}) + M_{t-1}^j}{P_t}\right] + \Pi_t,\ \Pi_t = \int_0^1 \left[\left(\frac{p_{jt}}{P_t}\right)^{1-\theta} - \frac{1}{\zeta_t}\frac{W_t}{P_t}\left(\frac{p_{jt}}{P_t}\right)^{-\theta}\right] Y_t.$$

This is justified since this is an analysis of the equilibrium and the budget equation above is an equilibrium conditions. Now use the cashless economy assumption and denote by b_t^j the amount of real bonds to deduce

$$C_t^j + b_t^j = \left(\frac{W_t}{P_t}\right) L_t^j + b_{t-1}^j(1+r_{t-1})\frac{1}{\pi_t} + \int_{s_t}\left[\left(\frac{p_{jt}}{P_t}\right)^{1-\theta} - \frac{1}{\zeta_t}\frac{W_t}{P_t}\left(\frac{p_{jt}}{P_t}\right)^{-\theta}\right] Y_t\, dj + \int_{s_t^c}\left[\left(\frac{p_{jt-1}}{P_t}\right)^{1-\theta} - \frac{1}{\zeta_t}\frac{W_t}{P_t}\left(\frac{p_{jt-1}}{P_t}\right)^{-\theta}\right] Y_t\, dj$$

By (9) one can simplify the budget constraint to

$$C_t^j + b_t^j = \left(\frac{W_t}{P_t}\right) L_t^j + b_{t-1}^j(1+r_{t-1})\frac{1}{\pi_t} + Y_t - \frac{1}{\zeta_t}\frac{W_t}{P_t} Y_t \left[\int_{s_t}\left(\frac{p_{jt}}{P_t}\right)^{-\theta} dj + \int_{s_t^c}\left(\frac{p_{jt-1}}{P_t}\right)^{-\theta} dj\right]. \tag{B.2}$$

To log linearize (B.2) use (9), (10a) and (10b) to conclude that

$$\begin{aligned}
\hat{c}_t^j + \hat{b}_t^j &= \frac{\theta-1}{\theta}\left[\hat{\ell}_t^j + \hat{w}_t - \hat{p}_t\right] + \frac{1}{\beta}\hat{b}_{t-1}^j + \hat{y}_t - \frac{\theta-1}{\theta}\left[-\hat{\zeta}_t + \hat{w}_t - \hat{p}_t + \hat{y}_t\right] \\
&= \frac{\theta-1}{\theta}\hat{\ell}_t^j + \frac{1}{\beta}\hat{b}_{t-1}^j + \frac{1}{\theta}\hat{y}_t + \frac{\theta-1}{\theta}\hat{\zeta}_t \quad \text{where } \hat{b}_t^j = \frac{b_t^j}{\bar{Y}}
\end{aligned}$$

From (5b) and (5b′) and from the production function we have the sequence

$$\hat{\varepsilon}_t^j = -\frac{\sigma}{\eta}\hat{c}_t^j + \frac{1}{\eta}(\hat{w}_t - \hat{p}_t) = -\frac{\sigma}{\eta}\hat{c}_t^j + \frac{1}{\eta}(\eta\hat{n}_t + \sigma\hat{y}_t)$$
$$= -\frac{\sigma}{\eta}\hat{c}_t^j + (\hat{y}_t - \hat{\zeta}_t) + \frac{\sigma}{\eta}\hat{y}_t = -\frac{\sigma}{\eta}\hat{c}_t^j - \hat{\zeta}_t + \left(1 + \frac{\sigma}{\eta}\right)\hat{y}_t$$

hence

$$\hat{c}_t^j + \hat{b}_t^j = \frac{\theta-1}{\theta}\left[\frac{-\sigma}{\eta}\hat{c}_t^j - \hat{\zeta}_t + \left(1 + \frac{\sigma}{\eta}\right)\hat{y}_t\right] + \frac{1}{\beta}\hat{b}_{t-1}^j + \frac{1}{\theta}\hat{y}_t + \frac{\theta-1}{\theta}\hat{\zeta}_t$$

Rearranging and solving for $\hat{b}_t^j$ the borrowing function is then

$$\hat{b}_t^j = \frac{1}{\beta}\hat{b}_{t-1}^j + \left[1 + \frac{\theta-1}{\theta}\frac{\sigma}{\eta}\right](\hat{y}_t - \hat{c}_t^j)$$

Return now to (B.1a) and (B.1b) and by matching coefficients deduce that

$$A_b^z = A_y^g\left[1 + \frac{\theta-1}{\theta}\frac{\sigma}{\eta}\right], \quad A_b^\zeta = 0, \quad A_b^b = \frac{1}{\beta} - A_y^b\left[1 + \frac{\theta-1}{\theta}\frac{\sigma}{\eta}\right], \quad A_b^g = -A_y^g\left[1 + \frac{\theta-1}{\theta}\frac{\sigma}{\eta}\right] \tag{B.3}$$

This last computation was based on a comparison of the following two functions:

$$\hat{c}_t^j = A_y^z Z_t + A_y^\zeta\hat{\zeta}_t + A_y^b\hat{b}_{t-1}^j + A_y^g g_t^j = A_y \bullet (Z_t, \hat{\zeta}_t, \hat{b}_{t-1}^j, g_t^j)$$
$$\hat{y}_t = A_y^z Z_t + A_y^\zeta\hat{\zeta}_t + A_y^b 0 + A_y^g Z_t = A_y \bullet (Z_t, \hat{\zeta}_t, 0, Z_t)$$

To see why, note that given the individual decision functions above, aggregate functions are deduced from them by market clearing conditions which are

$$\int_0^1 \hat{c}_t^j\, dj = \hat{y}_t, \int_0^1 \hat{b}_t^j\, dj = 0, \; Z_t = \int_0^1 \hat{g}_t^j\, dj, \quad \int_0^1 \hat{q}_{jt}^* = \frac{\omega}{1-\omega}\hat{\pi}_t$$

By (B.1a)–(B.1c)

$$\hat{y}_t = A_y^Z Z_t + A_y^\zeta\hat{\zeta}_t + A_y^b 0 + A_y^g Z_t \equiv A_y \bullet (Z_t, \hat{\zeta}_t, 0, Z_t) \tag{B.4a}$$

$$\hat{\pi}_t = A_\pi^Z Z_t + A_\pi^\zeta\hat{\zeta}_t + A_\pi^b 0 + A_\pi^g Z_t \equiv A_\pi \bullet (Z_t, \hat{\zeta}_t, 0, Z_t) \tag{B.4b}$$

$$\hat{q}_t = \frac{\omega}{1-\omega}\left[A_\pi^z(Z_t + A_\pi^\zeta\hat{\zeta}_t + A_\pi^b 0 + A_\pi^g Z_t\right] \equiv \frac{\omega}{1-\omega}A_\pi \bullet (Z_t, \hat{\zeta}_t, 0, Z_t). \tag{B.4c}$$

B.1. A note on simulations

Once matching of parameters is completed and the values of (A_y, A_π, A_b) are determined, the solutions (B.4a)–(B.4c) can be used to simulate the system of structural equations and law of motion. This is a simple procedure in which one uses (B.4a)–(B.4c) to select an initial condition given some $(Z_0, \hat{\zeta}_0)$. Next one simulates a system like (33a) and (33b) to obtain a sequence of $(Z_t, \hat{\zeta}_t)$ which is then inserted into (B.4a)–(B.4c) to compute the implied values of the aggregate endogenous variables.

The method outlined is simpler than the standard procedures used to simulate a Blanchard–Kahn type of a macro-system. However, a standard macro-simulation can be carried out with off-the-shelf programs for simulating forward looking system of difference equations. A standard procedure can be used once the constants (B_y, B_π) are computed using the parameters computed from the macro-economic equilibrium as outlined above. The reader can check that *the results are identically the same in the two methods*. Naturally, the simplicity of the first method results from the general equilibrium approach taken in the text to determine individual decision functions and their parameters in the log linearized economy as outlined in this appendix. However, if the micro-economic equilibrium becomes more complicated and entails, for example, an infinite number of state variables, such procedure may not be feasible.

References

Adam, K., 2007. Experimental evidence on the persistence of output and inflation. Economic Journal 117, 603–636.
Anufriev, M., Assenza, T., Hommes, C., Massaro, D., 2008. Interest rate rules with heterogeneous expectations. Working Paper. Department of Economics, University of Amsterdam.
Arifovic, J., Bullard, J., Kostyshyna, O., 2007. Social learning and monetary policy rules. Working Paper Series 2007-007. Federal Reserve Bank of St. Louis.
Ball, L.B., 1999. Policy rules for open economies. In: Taylor, J.B. (Ed.), Monetary Policy Rules, University of Chicago Press, Chicago, pp. 127–156. (Chapter 3).
Basu, S., 1996. Procyclical productivity: increasing returns or cyclical utilization? Quarterly Journal of Economics 111, 719–751.
Batchelor, R., Dua, P., 1991. Blue chip rationality tests. Journal of Money, Credit and Banking 23, 692–705.
Blanchard, O.J., Galí, J., 2010. Labor markets and monetary policy: a new Keynesian model with unemployment. American Economic Journal: Macroeconomics 2, 1–30.

Blanchard, O.J., Kahn, C.M., 1980. The solution of linear difference models under rational expectations. Econometrica 48, 1305–1311.
Branch, W.A., Evans, G.W., 2006. Intrinsic heterogeneity in expectation formation. Journal of Economic Theory 127, 264–295.
Branch, W.A., Evans, G.W., 2011. Monetary policy with heterogeneous expectations. Economic Theory 47, 365–393.
Branch, W.A., McGough, B., 2009. A new Keynesian model with heterogenous expectations. Journal of Economic Dynamics and Control 33, 1036–1051.
Branch, W.A., McGough, B., 2011. Business cycle amplification with heterogenous expectations. Economic Theory 47, 395–421.
Brazier, A., Harrison, R., King, M., Yates, T., 2008. The danger of inflating expectations of macroeconomic stability: heuristic switching in an overlapping generations monetary model. International Journal of Central Banking 4, 219–254.
Bullard, J., Mitra, K., 2002. Learning about monetary policy rules. Journal of Monetary Economics 49, 1005–1129.
Calvo, G.A., 1983. Staggered prices in a utility maximizing framework. Journal of Monetary Economics 12, 983–998.
De Grauwe, P., 2011. Animal spirits and monetary policy. Economic Theory 47, 423–457.
Eichenbaum, M., 1991. Real business-cycle theory, wisdom or whimsy? Journal of Economic Dynamics and Control 15, 607–626.
Evans, G., Honkapohja, S., 2003. Expectations and the stability problem for optimal monetary policies. Review of Economic Studies 70, 807–824.
Evans, G., Honkapohja, S., 2006. Monetary policy, expectations and commitment. Scandinavian Journal of Economics 108 (1), 15–38.
Frankel, J.A., Froot, K.A., 1990. Chartists, fundamentalists and the demand for dollars. In: Courakis, A.S., Taylor, M.P. (Eds.), Private Behavior and Government Policy in Interdependent Economies, Oxford University Press, New York., pp. 73–126.
Frankel, J.A., Rose, A.K., 1995. A survey of empirical research on nominal exchange rates. In: Grossman, G.M., Rogoff, K. (Eds.), Handbook of International Economics III, North Holland, Amsterdam., pp. 1689–1729 (Chapter 33).
Freedman, D., 1963. On the asymptotic behavior of bayes estimates in the discrete case I. Annals of Mathematical Statistics 34, 1386–1403.
Freedman, D., 1965. On the asymptotic behavior of bayes estimates in the discrete case II. Annals of Mathematical Statistics 36, 454–456.
Fuhrer, J.C., 1994. Optimal Monetary Policy and the Sacrifice Ratio. Boston Federal Reserve.
Galí, J., 2008. Monetary Policy, Inflation and the Business Cycle: An Introduction to the New Keynesian Framework. Princeton University Press, Princeton.
Gertler, M., Sala, L., Trigari, A., 2008. An estimated monetary DSGE model with unemployment and staggered nominal wage bargaining. Journal of Money, Credit and Banking 40, 1713–1764.
Howitt, P., 1992. Interest rate control and nonconvergence to rational expectations. Journal of Political Economy 100, 776–800.
Jin, H., 2007. Nominal interest rate rules under heterogenous beliefs. Rivista Internazionale Di Scienze Sociali CXV 3, 403–442.
Kandel, E., Pearson, N.D., 1995. Differential interpretation of public signals and trade in speculative markets. Journal of Political Economy 4, 831–872.
King, R.G., Rebelo, S.T., 1999. Resuscitating real business cycles. In: Taylor, J., Woodford, M. (Eds.), Handbook of Macroeconomics, vol. 1B. , North-Holland, Amsterdam., pp. 927–1007 (Chapter 14).
Kurz, M., 1994. On the structure and diversity of rational beliefs. Economic Theory 4, 877–900 (an edited version appears as Chapter 2 of Kurz, M. (Ed.), 1997).
Kurz, M. (Ed.), Studies in Economic Theory, vol. 6. , Springer-Verlag, Berlin, New York.
Kurz, M., 2008. Beauty contests under private information and diverse beliefs: how different? Journal of Mathematical Economics 44, 762–784.
Kurz, M., 2009. Rational diverse beliefs and economic volatility. In: Hens, T., Schenk-Hoppé, K.R. (Eds.), Handbook of Financial Markets: Dynamics and Evolution, North-Holland, Amsterdam., pp. 440–506 (Chapter 8).
Kurz, M., 2011. On the role of market belief in economic dynamics: an introduction. Economic Theory 47, 189–204.
Kurz, M., 2012. A new Keynesian model with diverse beliefs. Working Paper. Stanford University, February 27, 2012.
Kurz, M., Motolese, M., 2001. Endogenous uncertainty and market volatility. Economic Theory 17, 497–544.
Kurz, M., Motolese, M., 2011. Diverse beliefs and time variability or risk premia. Economic Theory 47, 293–335.
Kurz, M., Wu, H.M., 1996. Endogenous uncertainty in a general equilibrium model with price contingent contracts. Economic Theory 8, 461–488 (appears as Chapter 2 of Kurz, M. (Ed.), 1997).
Kurz, M., Jin, H., Motolese, M., 2005. The role of expectations in economic fluctuations and the efficacy of monetary policy. Journal of Economic Dynamics and Control 29, 2017–2065.
Lorenzoni, G., 2009. A theory of demand shocks. American Economic Review 99, 2050–2084.
Lucas Jr., R.E., 1987. Models of Business Cycles. Basil Blackwell, New York Yrjö Jahnsson Lectures, Section III, pp. 20–31.
Lucas Jr., R.E., 2003. Macroeconomic priorities. American Economic Review 4, 1–14.
Massaro, D., 2012. Heterogeneous expectations in monetary DSGE models. Mimeo. University of Amsterdam.
Milani, M., 2011. Expectation shocks and learning as drivers of the business cycle. Economic Journal 121, 379–401.
Motolese, M., 2001. Money non-neutrality in a rational belief equilibrium with financial assets. Economic Theory 18, 97. 16.
Motolese, M., 2003. Endogenous uncertainty and the non-neutrality of money. Economic Theory 21, 317–345.
Rudebusch, G., Svensson, L.E.O., 1999. Policy rules for inflation targeting. In: Taylor, J.B. (Ed.), Monetary Policy Rules, University of Chicago Press, Chicago., pp. 203–262 (Chapter 5).
Summers, L.H., 1986. Some skeptical observations on real business cycle theory. Federal Reserve Bank of Minneapolis Quarterly Review 10, 23–27.
Takagi, S., 1991. Exchange rate expectations: a survey of survey studies. International Monetary Fund Staff Papers 38, 156–183.
Taylor, B., 1979. Estimation and control of a macroeconomic model with rational expectations. Econometrica 47, 1267–1286.
Walsh, C.E., 2010. Monetary Theory and Policy, Third edition The MIT Press, Massachusetts.
Woodford, M., 2003. Interest and Prices: Foundations of a Theory of Monetary Policy. Princeton University Press, Princeton.
Woodford, M., 2010. Robustly optimal monetary policy with near rational expectations. American Economic Review 100, 274–303.